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Title:
On the qualitative theory of second order elliptic operators.

ON THE QUALITATIVE THEORY OF SECOND ORDER ELLIPTIC OPERATORS

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December 2001

A dissertation submitted to the University of Bristol in accordance with the requirements of the degree of Doctor of Philosophy in the Faculty of Science

Abstract

In this research we are concerned with some aspects of the qualitative theory of second order elliptic operators. In particular we study the problem of strong uniqueness in L^p-spaces for operators of this type, with both the cases of finite and infinite dimensional state spaces being investigated. In the first case we study the classical Dirichlet operator perturbed by singular lower order terms. We reveal sufficient conditions on the coefficients of the differential operator, which ensure that the latter has a unique extension generating a C_0-semigroup on L^p. It is worth mentioning that the property of strong uniqueness for the perturbed Dirichlet operator holds even if the associated quadratic forms are not sectorial. In the case of infinite dimensional state space we establish the uniqueness of the Dirichlet operator with variable diffusion coefficients. The approaches, used in both cases, are based on a priori estimates of solutions of the corresponding elliptic and parabolic equations.

The final part of this work is devoted to the studying elliptic and parabolic equations with measurable (singular) lower order coefficients. We show that the parabolic equation in question has a unique weak fundamental solution that enjoys global in time Gaussian upper and lower bounds. These estimates are applied to the problem of existence and non-existence of positive weak solutions for a class of semi-linear elliptic inequalities. We also provide examples that prove sharpness of the existence and non-existence results.

Acknowledgments

In the first place I would like to thank Vitali Liskevich, my teacher and main advisor, who started me on the topic of this research and provided a crucial support and encouragement in the course of this project.

I am also largely indebted to Zeev Sobol who spared me a substantial amount of his time and advice. Without his participation the results presented in Chapter 4 would be of much less value and interest.

My special gratitude is to Michael Röckner for helpful discussions of different aspects of finite and infinite dimensional analysis, Vladimir Kondratiev for a series of instructive lectures on the classical theory of weak solutions, and Michiel van den Berg, my second advisor, for the valuable feedback and essential support for my research.

I am very grateful to my closest relatives and friends for their vital moral support over the past three years.

Last but not the least, I am thankful to the University of Bristol and Committee of Vice-Chancellors and Principals for the financial support of this Ph.D. project, and the School of Mathematics for friendly, stimulating environment and technical support.

Author's Declaration

I declare that the work in this thesis was carried out in accordance with the Regulations of the University of Bristol. The work is original except where indicated by special reference in the text and no part of the dissertation has been submitted for any other degree. Any views expressed in the dissertation are those of the author and in no way represent those of the University of Bristol. The thesis has not been presented to any other University for examination either in the United Kingdom or overseas.

Oleksiy Us

Date: 9 October 2001

Contents

Chapter 1

Introduction

In this research we are concerned with some aspects of the qualitative theory of second order elliptic operators, related to the formal differential expression

$$\Lambda u := \sum_{k,j} -\nabla_k^*(a_{kj}\nabla_j u) - \sum_j b_j \nabla_j u - qu, \tag{1.1}$$

where ∇_k^* stands for the formal adjoint to ∇_k. We concentrate on the corresponding elliptic and parabolic equations with measurable, generally singular coefficients. Such equations appear in various models describing diffusion in continuous media: fluid mechanics, heat and mass transfer, quantum and statistical physics, mathematical economics and biology; as well as in problems of differential geometry, potential theory and stochastic processes. Singular coefficients appear in the study of anisotropic inhomogeneous media, strong flows and non-regular absorption. In this work we use an approach, based on methods of the theory of strongly continuous semigroups of operators on Banach spaces.

We begin by explaining, in an informal way, how semigroups of operators arise in mathematical models.

It is natural to describe the evolution of a continuous time system by the initial value problem

$$\partial_t u(t) = F(t, u(t)), \quad t > 0, \quad u(0) = f,$$

where $t \geq 0$, $u(t)$ and $u(0) = f$ denote the time, the state of the system at time t and the initial state of the system, respectively; the function F describes the dynamics of the system. In order to make sense of the derivative $\partial_t u$, we assume that u varies in a Banach space X. In general F need be neither autonomous nor linear in u. It is standard practice in applications, however, to approximate F by a time-independent linear function. Thus we obtain the following Cauchy problem:

$$\partial_t u(t) = \mathcal{A}u(t), \quad t > 0, \quad u(0) = f, \tag{1.2}$$

where $\mathcal{A}$ is a linear operator in X.

Problem (1.2) is presumed to describe the evolution of a system. Therefore we expect the following properties to be fulfilled:

(i) a solution to (1.2) exists (otherwise there is no description of the evolution, so the model does not work);

(ii) the solution is unique (the outcome should be uniquely determined by the initial state and the dynamics);

(iii) the solution depends on the initial data and time continuously (otherwise the model could not be tested by an experiment, since unobservable changes in the initial data and time would drastically change the outcome).

Conditions (i)-(iii) above form the notion of well-posedness of the problem. Let (1.2) be well-posed.

For $t > 0$ let $T(t)$ stand for the mapping of the initial data f into the state $u(t)$ of the system at time t, i.e. $u(t) = T(t)u(0)$. It follows from the uniqueness property that $u(t+s) = T(t)u(s)$, hence, we derive the semigroup identity $T(t+s) = T(t)T(s)$. The equality $u(t) = T(0)u(t)$ leads to a natural definition $T(0) :=$ Id, where Id stands for the identity operator. Property (iii) implies that the operator-valued function $T(\cdot)$ is continuous in some topology. It is clear that the operator $T(t)$ is linear for all $t \geq 0$ as is $\mathcal{A}$.

The theory of strongly continuous (C_0-) semigroups is nowadays a well developed branch of functional analysis with numerous applications in partial differential equations, probability theory and many other areas of modern mathematics. We refer the reader to the books by E. B. Davies [14], A. Pazy [62], J. Goldstein [29], K.-J. Engel and R. Nagel [23] and classical works by E. Hille and R.S. Philips [32] and K. Yosida [79] which are excellent sources on the theory of semigroups of linear operators. This abstract theory also proves to be a powerful tool for investigating parabolic and elliptic equations.

The classical theory of second order elliptic and parabolic equations deals with equations in non-divergent form, i.e. in (1.1) the differential expression Λ is of the form

$$\Lambda u = \sum_{k,j} a_{kj} \nabla^2_{kj} u - \sum_j \hat{b}_j \nabla_j u - Vu.$$

It was J. Schauder who noticed that elliptic equations with Hölder continuous $a_{kj}, \hat{b}_j, V$ can be treated as perturbations of equations with constant coefficients. Fundamental to this approach are a priori *interior, boundary and global* estimates of solutions. For detailed exposition and extension of the Schauder theory of classical solutions see [28, Ch. 6]. A traditional approach to the solvability of the Cauchy problem (1.2) for the non-divergent parabolic equation involves C_0-semigroups. The main idea is to show that the operator $\mathcal{A}$ is the generator of a C_0-semigroup on a specific Banach space X, which is in most cases one of the Lebesgue spaces L^p, spaces of bounded continuous or Hölder continuous functions. The most frequently employed tools are the Hille-Yosida-Philips or Solomyak-Yosida theorems (see e.g. [62, Ch. 1, Ch. 2]). In the former case one needs to check that $\mathcal{A}$ is densely defined, closed and all positive integer powers of its resolvent satisfy certain upper bounds. The last condition can be quite difficult to verify, since one may require the Schauder or Agmon-Douglis-Nirenberg a priori estimates of solutions to corresponding elliptic problems (see [28], [57, Ch.3]; see also [40]). Problem (1.2) then has a unique solution $u(t) = T(t)f$, $t \geq 0$, for all $f \in \mathcal{D}(\mathcal{A})$. The domain $\mathcal{D}(\mathcal{A})$ of the generator is usually described explicitly. In the latter case (when the conditions of the Solomyak-Yosida theorem are fulfilled) the operator generates an analytic semigroup on X, so that the unique solution to (1.2) exists for all initial data $f \in X$. One can see that the above approaches require quite restrictive assumptions on the coefficients of Λ. In order to relax these conditions one has to extend the notion of solution. This is the reason why "good solutions", which can be obtained as limits (in a proper sense) of classical solutions, were introduced and the corresponding theory was developed. However, N. Nadirashvili [60] discovered that a "good solution" to the Dirichlet problem fails the uniqueness property, which significantly diminishes the value of the theory for applications.

On the other hand, for divergence-type equations with measurable coefficients one manages to develop a rich theory, using the the so-called weak solutions. The idea behind this notion is to replace the pointwise differential equation $\Lambda u = f$ by the integral equality $\int u\Lambda^* \varphi dx = \int f\varphi dx$ for a certain class of test functions φ, where Λ^* denotes the formal adjoint to Λ. The concept of weak solution was the starting point for development of Hilbert space techniques in partial differential equations. These methods exploit the well-known Lax-Milgram theorem (see e.g. [21]) and are based on the observation that the left-hand side of the above integral equality defines a sesquilinear form on some Sobolev space. Under rather mild restrictions on the matrix a, one can guarantee existence of a weak solution. By the First Representation Theorem, which follows from the Lax-Milgram theorem, there exists a unique m-sectorial operator $-\mathcal{A}$ associated with a closed, densely defined, sectorial form. The operator $\mathcal{A}$ is known to generate an analytic semigroup on some Hilbert space, so the unique solution to the Cauchy problem (1.2) is given by $u(t) = \exp(t\mathcal{A})f$, $t \geq 0$. In general the domain $\mathcal{D}(\mathcal{A})$ of the generator $\mathcal{A}$ can only be described implicitly. However, if the coefficients of Λ are good enough, then $\mathcal{A}$ extends the initial operator defined by means of Λ on the set of smooth functions. The solution u to (1.2) is a weak solution.

Next we describe in details the questions of the qualitative theory of second order elliptic operators, which constitute the subject of the present work. The first problem is concerned with a differential operator $\mathcal{H}$ in $L^p(\mathbb{R}^d, \rho dx) =: L^p$, defined by

$$\mathcal{H} = -\Delta - \beta \cdot \nabla + b \cdot \nabla + q \tag{1.3}$$

on the set $C_0^\infty(\mathbb{R}^d)$ of smooth compactly supported functions on $\mathbb{R}^d$. Here β is the logarithmic derivative of the Radon measure ρdx, b is a measurable vector field and q is a complex-valued measurable potential. Operators of this type appear as Hamiltonians in formulation of dynamics via energy forms (see [1, 3]). Under certain conditions on β, b and q one can construct an operator $-\mathcal{A}$, such that $\mathcal{A}$ generates a C_0-semigroup on L^p and $-\mathcal{A} \supset \mathcal{H}$. In order to ensure the well-posedness of the Cauchy problem, related to the operator $\mathcal{H}$, one needs to know that such an extension is unique. This problem is referred to as the strong uniqueness problem and in the case $p = 2$, $b = 0$ and $q = \operatorname{Re} q$, is equivalent to the essential self-adjointness. One of the aims of the present work is to reveal sufficient conditions on the coefficients β, b and q which guarantee the property of strong uniqueness for the operator $\mathcal{H}$.

It is readily seen that the closure of the sesquilinear form

$$\mathcal{E}(u, v) = \langle \nabla u, \nabla v \rangle, \quad u, v \in C_0^1(\mathbb{R}^d), \tag{1.4}$$

is a Dirichlet form in the sense of [15, Ch. 1]. The associated operator is denoted by $\mathcal{L}$ and called the Dirichlet operator corresponding to the measure $\rho\, dx$ (for the extensive treatment of symmetric and non-symmetric Dirichlet forms and operators see [58]). It is well-known and follows from the Beurling-Deny criteria (see e.g. [27]; see also subsection 2.2.3) that $-\mathcal{L}$ generates a sub-Markovian semigroup on L^2. The semigroup $\exp(-t\mathcal{L})$ gives rise to a family of positive contraction semigroups $\exp(-t\mathcal{L}_p)$, $p \geq 1$, by

$$\exp(-t\mathcal{L}_p) := \left(\exp(-t\mathcal{L}) \restriction_{L^1 \cap L^\infty}\right)^{\sim}_{L^p \to L^p},$$

where $\sim$ and $\restriction$ stand for the closure and the restriction signs respectively. If $p < \infty$ the semigroups $\exp(-t\mathcal{L}_p)$ are strongly continuous. We also note that for $p \in (1, \infty)$ the proof of the C_0-property is an immediate consequence of the Riesz-Thorin interpolation theorem (see e.g. [15, Th. 1.1.5]), whereas the case $p = 1$ requires a more delicate argument (for details see e.g. [15, Th. 1.4.1]; see also [11]).

The operator $\mathcal{L}_p$ is called the Dirichlet operator in L^p. Below we give several reasons for developing L^p-theory of Dirichlet operators.

Firstly, it is meaningful to investigate the well-posedness of the Cauchy problem with initial data from L^p, not only from L^2. Secondly, we observe that conditions, imposed on the weight ρ, do not guarantee that the form domain $\mathcal{D}(\mathcal{L}^{\frac{1}{2}})$ is a subspace of L^r for some $r > 2$ since, in general, the Sobolev embedding theorem is not valid in weighted spaces. Under these circumstances existence of a consistent C_0-semigroup on L^p provides information about the regularity of the solution to the corresponding Cauchy problem in L^2 with initial data $f \in L^2 \cap L^p$. We think of semigroups $\exp(-t\mathcal{L}_p)$, $1 \leq p < \infty$, as describing the evolution of a "free" physical system. In particular, if $\rho \equiv 1$, these are the semigroups, generated by the free Laplacian, in which case the extension of $\Delta \restriction_{C_0^\infty(\mathbb{R}^d)}$, generating a C_0-semigroup, is known to be unique. The corresponding Cauchy problem then has a unique solution $u_p(t) = \exp(t\Delta_p)f$, $t \geq 0$, in L^p for all $f \in L^p$. If however, $\rho \not\equiv 1$, then it is, in general, no longer the case (see e.g. [19]), and a natural question arises: what are the restrictions on the weight ρ, which guarantee that $-\mathcal{L}_p$ is a unique extension of $\Delta + \beta \cdot \nabla \restriction_{C_0^\infty(\mathbb{R}^d)}$ with this property? If we assume, in addition, that $\beta \in L^p_{loc}$, then $C_0^\infty(\mathbb{R}^d) \subset \mathcal{D}(\mathcal{L}_p)$ and $\mathcal{L}_p = -\Delta - \beta \cdot \nabla$ on $C_0^\infty(\mathbb{R}^d)$. We also note that the strong uniqueness in L^p is known to be equivalent to the fact that $C_0^\infty(\mathbb{R}^d)$ is a core of the operator $\mathcal{L}_p$ (see e.g. [61, Th. AII, 1.33]).

The problem of strong uniqueness has a long history. It goes back to [3] and has been intensively studied in recent years ([6],[10],[19],[20],[45]; also see [19] for non-uniqueness results and for extensive discussion of various types of the uniqueness problem). The examples (see [19]) show that, when investigating the strong uniqueness for the operator $\mathcal{L}_p$, the condition $\beta \in L^{2p}_{loc}$ cannot be replaced by $\beta \in L^{2p-\varepsilon}_{loc}$ for any $\varepsilon > 0$, however the question whether this condition is sufficient remains open. As a particular case of the main results in Chapter 3 we prove a criterion of strong uniqueness for the operator $\mathcal{L}_p \restriction_{C_0^\infty(\mathbb{R}^d)}$ in L^p, $3/2 < p \leq 2$, under the assumption $\beta \in L^{2p}_{loc}$ and an additional local condition in the form of a weighted Hardy-type inequality outside a ball in $\mathbb{R}^d$. The latter replaces the requirement, imposed in earlier researches on strong uniqueness (see e.g. [46]), that the measure ρdx is finite.

In order to establish the strong uniqueness for the operator $\mathcal{L}_p$ in L^p in the case $1 < p \leq 3/2$ we impose a more restrictive condition on the logarithmic derivative β. Namely, we assume that $\beta \in L^{\frac{2}{2-p}}_{loc}$. We believe that this condition is technical, although no better results are known thus far.

Next we discuss the perturbation theory for the operator $\mathcal{L}_p$. Let b and q be as in (1.3). Now the Cauchy problem, related to (1.3), can be formally written as

$$\frac{\partial v}{\partial t} + (\mathcal{L} + b \cdot \nabla + q)v = 0, \ v(0) = f. \tag{1.5}$$

We write $q = V^+ - V^- + iW$. One can see from [47] that if $b = 0$ and $V^- = 0$, then the problem (1.5) is well-posed in L^p for *all* $p \in [1, \infty)$. However, this is no longer true if $|b|^2$ and V^- are only assumed to be form-bounded with respect to $\mathcal{L}$ (see e.g. [8], [52]). In Chapter 3 we construct the generator $-H_p$ of a C_0-semigroup on L^p in a closed interval I of the L^p-scale, containing 2, with I depending on the form-bound. This extends the corresponding result from [8]. We stress that the sesquilinear form, related to $\mathcal{H}$, in general, need not be sectorial since the potential W is not assumed to be form-bounded with respect to $\mathcal{L}$. Thus the Cauchy problem (1.5) is well-posed for all $f \in \mathcal{D}(H_p)$. If, in addition, we have β, b, $q \in L^p_{loc}$, then $\mathcal{D}(H_p) \supset C_0^\infty(\mathbb{R}^d)$ and $H_p = \mathcal{H}$ on $C_0^\infty(\mathbb{R}^d)$.

Having constructed H_p we turn to studying the strong uniqueness problem for this operator. In the case $p \in \operatorname{Int} I \cap (1, 2]$ we keep the assumptions on the logarithmic derivative β unchanged and reveal sufficient conditions on the drift coefficient b and potential q which ensure that the property of strong uniqueness for the operator H_p holds, i.e. that $-H_p$ is the only extension of $-\mathcal{H}$ that generates a C_0-semigroup. Hence, we extend the main result from [55] to the L^p-setting.

The problem of strong uniqueness for Schrödinger operators, i.e. the case $\rho \equiv 1$

and $b = 0$, was addressed in many researches. A review of earlier results on the essential self-adjointness can be found in [64]. The strong uniqueness problem for Schrödinger operators in L^p was studied in [8] (see also references therein). The present investigation is carried out in two steps. First we develop an approach analogous to that used by C. Simader [70] and H. Brezis [12] for Schrödinger operators in L^2, reducing ("localising") the problem to the one for a degenerate operator with coefficients vanishing outside a ball in $\mathbb{R}^d$. The conditional theorem, we prove, states that the operator H_p is unique, provided the degenerate operator has this property, and extends the result from [70]. In order to complete the proof of the strong uniqueness for H_p we split the degenerate operator into "no-potential" and "potential" parts. We treat the "no-potential" part by employing the method of a priori estimates, developed in [51], and apply an abstract perturbation argument from [52] to studying the "potential" component.

The case $p > 2$ turns out to be more complicated. At present the uniqueness result, we managed to prove, involves some implicit conditions on the coefficients of $\mathcal{H}$ which may be hard to verify. The author believes that in order to treat this problem a new approach needs to be developed, and hopes to return to this task in the future.

In the second part of this work we deal with Dirichlet forms and operators in spaces of functions of infinitely many variables. The topic attracts much attention among the researchers because of the intimate relation between infinite dimensional Dirichlet forms and operators, theory of Markov processes, stochastic analysis and quantum field theory (see e.g. [11, 58, 66] and references therein). Below we give a more detailed comment on the subject.

It is well-known that one cannot define a standard Lebesgue-type measure in an infinite dimensional space (see e.g. [41]). There is, therefore, no canonical duality between spaces of test and generalised functions, which has proved to be so fruitful for the theory of partial differential equations in finite dimensions. It is Gaussian measures in infinite dimensional vector spaces that play fundamental role in the analysis of functions of infinitely many variables. In order to construct a Gaussian measure ν in a real separable Hilbert space $\mathcal{X}$ one can employ the abstract Wiener space approach developed by L. Gross ([30]). It is a remarkable fact that for ν to be σ-additive it needs to be defined on a wider space Y' that appears to be the completion of $\mathcal{X}$ with respect to a weaker norm. In a more general setting one usually has a rigging $Y' \supset \mathcal{X} \supset Y$ of a real separable Hilbert space $\mathcal{X}$ by a locally convex complete real vector space Y and its dual Y', and ν is a Borel (not necessarily Gaussian) probability measure on Y'. Under certain conditions on the measure ν the quadratic form

$$\mathcal{E}_\nu[u] = \int\limits_{Y'} |\nabla u|^2 d\nu,$$

defined on smooth finitely based functions on Y', is closable and its closure is a Dirichlet form. Traditionally we preserve the notation $\mathcal{E}_\nu$ for the closure unless it leads to confusion. The associated operator $\mathcal{L}_\nu$ is called the Dirichlet operator corresponding to the measure ν. The semigroup, generated by $-\mathcal{L}_\nu$, turns out to be sub-Markovian. If we assume, in addition, that ν is a Radon measure and the space Y' is Souslin, then the semigroup $\exp(-t\mathcal{L}_\nu)$ is associated with a homogeneous Markov process $\zeta(t)$, $t \geq 0$, with invariant measure ν and infinitesimal operator $\mathcal{L}_\nu$. The process $\zeta(t)$, $t \geq 0$, appears to solve, in the weak sense, the stochastic differential equation

$$d\zeta(t) = \beta^\nu(\zeta(t))dt + dw(t), \tag{1.6}$$

where $w(t)/\sqrt{2}$ is the standard Wiener process on $\mathcal{X}$ and β^ν is the logarithmic derivative of the measure ν (see [2]). To summarize, the Dirichlet form $\mathcal{E}_\nu$ defines a self-adjoint operator which generates a symmetric Markov process $\zeta(t)$ satisfying stochastic differential equation (1.6). Besides their intrinsic probabilistic and functional analytic importance, such processes and the corresponding potential theory are also of interest from the viewpoint of quantum mechanics. Dirichlet operators and the associated diffusion processes are related to quantum fields, e.g. in the case of certain measures ν on $Y' = S'(\mathbb{R}^d)$, where $S'(\mathbb{R}^d)$ stands for the Schwartz space of tempered distributions on $\mathbb{R}^d$.

In the present research we study infinite dimensional operators of the form

$$\hat{\mathcal{L}}u = -\sum_{k,j} a_{kj}\frac{\partial^2 u}{\partial x_j \partial x_k} - \sum_{k,j} \frac{\partial a_{kj}}{\partial x_k}\frac{\partial u}{\partial x_j} - \sum_{k,j} \beta_k^\mu a_{kj}\frac{\partial u}{\partial x_j},$$

where $u \in \mathcal{F}C_b^\infty$, i.e. the set of smooth finitely based functions on a locally convex vector space X, $a = (a_{kj})_{k,j\geq 1}$ is a symmetric positive definite diffusion matrix and $\beta^\mu := (\beta_k^\mu)_{k\geq 1}$ is the logarithmic derivative of a given probability measure μ on X. Under certain conditions on a and β^μ, specified below, the form

$$\mathcal{E}(u,v) = \sum_{k,j}\int\limits_X a_{kj}\frac{\partial u}{\partial x_k}\frac{\partial v}{\partial x_j}d\mu = \int\limits_X \hat{\mathcal{L}}u\, v d\mu, \quad u,v \in \mathcal{F}C_b^\infty.$$

is closable and its closure is clearly a Dirichlet form, so the associated operator $\mathcal{L}$ (which is the Friedrichs extension of $\hat{\mathcal{L}}$ in $L^2(X,\mu)$) generates a sub-Markovian semigroup $e^{-t\mathcal{L}}$. Thus $e^{t\mathcal{L}} \restriction_{L^\infty(X,\mu)}$ extends to a C_0-semigroup on $L^p(X,\mu)$ for every $p \in [1,\infty)$, with the generator $-\mathcal{L}_p$. The operator $\mathcal{L}_p$ is called the infinite dimensional Dirichlet operator in $L^p(X,\mu)$. Under appropriate conditions on a and β^μ we have $\mathcal{L}_p \supset \hat{\mathcal{L}}$. Our aim is to give sufficient conditions on a_{kj} and the "large" part of the logarithmic derivative (see section 4.1), implying that $\mathcal{L}_p$ is the *only* such extension. In the case $p = 2$ this is equivalent to the essential self-adjointness of $\hat{\mathcal{L}}$ and is known to be of importance for the investigation of spectral properties of $\mathcal{L}$, as well as stochastic dynamics in some lattice systems (see [6]).

The problem of strong uniqueness for the Dirichlet operators over infinite dimensional state space has been studied intensively in recent years (see, [4, 5, 6, 16, 51] for the case $p = 2$ and $a_{kj} = \delta_{kj}$, [19, 48] for $p \geq 1, a_{kj} = \delta_{kj}$, [37] for variable a_{kj} if $p = 2$, and [49] for arbitrary p). In [48] an approach was developed to combine the conditions on the logarithmic derivative from [6] and [51]. However, due to technical difficulties certain restrictions on the "large" part were imposed. The present study is an extension and generalisation of Theorem 1 in [48] in several directions. Firstly, we consider variable diffusion coefficients a_{kj}; in addition, the matrix a is not supposed to be uniformly bounded and uniformly elliptic. Secondly, we remove the said restrictions on the logarithmic derivative (see condition (ii(b)) of Theorem 4.1.2 in comparison with condition (iv) of [48, Theorem 1]). This was possible due to a new method of obtaining smoothness and dimension independent estimates for gradients of smooth approximating solutions. In addition, in Theorem 4.1.2 we correct the interval of strong uniqueness, which was stated wrongly in [48, Theorem 3]. Apart from the greater generality of the results in the present work, we simplify the framework in [49] in order to make the conditions imposed more transparent. We also include an example which could not be treated by previous uniqueness results.

In the final part of the research we are concerned with two aspects of the qualitative theory of parabolic and elliptic equations in the finite dimensional vector space, namely

(i) existence and uniqueness of fundamental solutions of parabolic equations, related to differential expression Λ, and validity of two-sided Gaussian estimates for the heat kernels;

(ii) existence and non-existence of positive weak solutions for a class of semi-linear inequalities.

In this work we use the results of part (i) as a tool to treat problem (ii). However, we stress that the problem of validity of Gaussian bounds on heat kernels of parabolic equations is also of independent interest. Below we give a detailed comment on both issues.

We study the inequality

$$\Lambda u + u^p \leq 0 \quad \text{in} \quad \Omega, \tag{1.7}$$

where Λ is given by (1.1) (with $\nabla_k^* = -\nabla_k$, a symmetric, uniformly elliptic matrix a and measurable b, V) and $p > 1$. We assume that $\Omega = K^c$, where K^c stands for the complement of a compact set K in $\mathbb{R}^d$, $d \geq 3$. with . Sometimes Ω is called an exterior domain.

The theory of semi-linear inequalities in unbounded domains is being extensively developed because of its applications in mathematical physics and rich mathematical structure. In particular, the question whether inequalities of type (1.7) have or do not have positive weak solutions draws much attention among the experts in the field (see e.g. [9, 35, 38, 43, 59] and references therein).

Estimates of the type

$$c_1|x-y|^{2-d} \leq G(x,y) \leq c_2|x-y|^{2-d} \quad x, y \in \mathbb{R}^d,\, x \neq y,\; c_1, c_2 > 0, \tag{1.8}$$

where G is the fundamental solution of the equation $\Lambda v = 0$ in $\mathbb{R}^d$, become crucial when treating (1.7). It is well-known that bounds (1.8) hold if, for example, the fundamental solution $r = r(t,x,y)$ of the equation

$$\partial_t u - \Lambda u = 0, \quad t > 0,\; x, y \in \mathbb{R}^d, \tag{1.9}$$

enjoys global in time Gaussian upper and lower bounds, i.e. there exist positive constants $\gamma, \bar{\gamma}$, C_γ, $C_{\bar{\gamma}}$ such that

$$C_{\bar{\gamma}} t^{-d/2} e^{-\frac{|x-y|^2}{4\bar{\gamma}t}} \leq r(t,x,y) \leq C_\gamma t^{-d/2} e^{-\frac{|x-y|^2}{4\gamma t}}, \; t > 0,\, x, y \in \mathbb{R}^d. \tag{1.10}$$

It is for that reason that we study equation (1.9) with time-independent coefficients. If (1.10) are only valid for $t \in (0,T]$ for some $T > 0$, then we call the bounds local.

In [80, 81, 82] Q. Zhang considered the problem of validity of global and local Gaussian bounds on the heat kernel r of equation (1.9). Due to a gap in the limiting arguments (see [80, p. 67], [81, p. 388], [82, p. 988]) the results stated are valid only under additional restrictions on the lower order terms or under the assumption that r exists. However, for most of the applications, including the investigation of semi-linear inequality (1.7), it is vital to know both existence of fundamental solutions and validity of two-sided Gaussian estimates. For this reason we begin by studying equation (1.9).

There is extensive literature on heat kernel bounds for equations of type (1.9) with singular coefficients. It was D. Aronson [7] who first proved the existence of the fundamental solution of (1.9) with time-dependent coefficients and validity of *local* Gaussian estimates. At this point it is appropriate to recall the conditions of Aronson's theorem. Let a be a strictly elliptic matrix of bounded measurable coefficients and there is a ball $B_R \subset \mathbb{R}^d$ such that b, V are uniformly bounded outside the ball, $|b| \in L^q((0,T); L^p(B_R))$ with $p, q > 2$, $d/2p + 1/q < 1/2$, and $V \in L^q((0,T); L^p(B_R))$ with $p, q > 1$, $d/2p + 1/q < 1$. It is worth mentioning, however, that global Gaussian bounds cannot be obtained under Aronson's conditions.

Since [7], substantial progress has been made in understanding properties of heat kernels for particular cases of (1.9). In [71] B. Simon proved Gaussian bounds for the fundamental solution r in the case $a_{kj} = \delta_{kj}$, $b = 0$, under the assumption that V belongs to the so-called *Kato* class K_d. It is easy to check that time-independent potentials, satisfying Aronson's conditions, belong to K_d. The following result is due to J. Voigt: the condition that V belongs to the *enlarged* Kato class is a *necessary* condition for local Gaussian bounds to hold (some details on Kato classes can be found in [71, 77, 13]; see also subsection 2.6.2).

The corresponding Kato class for the drift coefficient is K_{d+1}. In [69] Gaussian estimates of the fundamental solution of (1.9) in the case $a_{kj} = \delta_{kj}$, $V = 0$, were obtained under the assumption $|b| \in K_{d+1}$. The notion of Kato classes was then extended to include the time-dependent case (see [67, 80, 82, 53]) and Gaussian estimates for the corresponding evolution families were derived under certain additional conditions (see [80, 81, 82, 53]).

In the present research we prove the existence of the heat kernel of (1.9) by combining a semigroup approach, a priori estimates, obtained in [80, 81, 82], and additional estimates of the gradients of weak solutions to the corresponding Cauchy problems. First we recall that the fundamental solution of equation (5.1) with bounded b and V exists and, due to [80, 81, 82], enjoys Gaussian estimates, with the estimates independent of L^∞-norms of b and V. The latter enables us to obtain the weak fundamental solution of (1.9) in the general case as a limit, in a proper sense, of the heat kernels corresponding to the operators with bounded coefficients. We then show that the limit enjoys Gaussian lower and upper bounds. Uniqueness of the fundamental solution follows by a standard argument. In order to obtain global Gaussian estimates we need to impose restrictions on the behaviour of the lower order terms at infinity, namely, we assume that b and V belong to the corresponding classes of *Green bounded* potentials (see Theorem 5.1.2). The fact that the coefficients of (1.9) are time-independent is essential for the proof of this result. The question whether the respective statements from [80, 81, 82] are valid in the case of time-dependent coefficients remains open.

Having established (1.10), and thus (1.8), we turn to studying inequality (1.7). Estimates (1.8) provide a powerful tool for treating semi-linear inequalities of type (1.7) in exterior domains. Our aim is to investigate for which $p > 1$ inequality (1.7) has, or does not have, positive weak solutions.

In [36] the authors studied inequality (1.7) under quite general conditions on the potential V, assuming that $b = 0$. In this work we are mainly interested in revealing conditions on the drift coefficient b, which ensure the existence or non-existence of positive weak solutions of inequality (1.7). The examples provided show that the behaviour of the first order terms makes a strong impact on existence of positive weak solutions of (1.7). The examples also prove sharpness of the result.

The method is based on estimates (1.10), and is clearly dependent on the validity of global Gaussian bounds for the heat kernel of (1.9). The author is not aware of other methods to obtain classical estimates of type (1.8) in the case when the lower order coefficients of Λ are only assumed to be Green bounded. The Harnack inequality for positive weak solutions of the equation $\Lambda v = 0$ in bounded domains also plays an important role in our proofs. It was shown in [26, 80, 82] that local Gaussian bounds for the fundamental solution of (1.9) are sufficient for the Harnack inequality to hold. Q. Zhang [80, 82] proved the Harnack inequality in the case $|b| \in K_{d+1}$, $V \in K_d$, under an additional assumption that the entries of the matrix a are uniformly Hölder continuous. In [42] the Harnack inequality was established by a different method for $|b|^2, V \in K_d$, without any extra regularity conditions being imposed on a.

The dissertation is organised as follows. In Chapter 2 we collect some known facts of the theory of sectorial forms (section 2.1), strongly continuous semigroups (sections 2.2 and 2.3) and partial differential equations (section 2.6). We also give a brief overview of some classical results on the problem of strong uniqueness for Schrödinger operators (section 2.4). In order to introduce the reader to some of the methods employed we apply these techniques to prove several well-known results (see Theorems 2.4.17 and 2.6.4). We also collect some important topological concepts which appear in Chapter 4 (section 2.5). The problem of strong uniqueness for the perturbed Dirichlet operator in $L^p(\mathbb{R}^d)$ is treated in Chapter 3, whereas in Chapter 4 we study the uniqueness problem for infinite dimensional Dirichlet operators with variable diffusion coefficients. Existence and uniqueness of the heat kernel of equation (1.9) and validity of global Gaussian bounds are established in Chapter 5. These estimates are then employed to investigate the problem of existence and non-existence of positive weak solutions to semi-linear inequalities of type (1.7).

Chapter 2

Background Material

2.1 Sectorial Forms

2.1.1 Basic Notions and Properties

In this section we collect some important notions and results of the theory of sesquilinear forms in Hilbert spaces. In the first two subsections we mainly follow [33, Ch. VI].

Let $\mathcal{X}$ be a Hilbert space with an inner product $\langle\cdot, \cdot\rangle$ and $\mathcal{D}$ a linear subspace of $\mathcal{X}$. A mapping $\tau : \mathcal{D} \times \mathcal{D} \mapsto \mathbb{C}$ is called a *sesquilinear form* if $\tau(u, v)$ is linear in u for each fixed $v \in \mathcal{D}$, and antilinear in v for each fixed $u \in \mathcal{D}$. The set $\mathcal{D}$ is called the *domain* of the form τ and is denoted by $\mathcal{D}(\tau)$. The form τ is said to be *densely defined* if $\mathcal{D}(\tau)$ is dense in $\mathcal{X}$.

The mapping $\tau' : \mathcal{D}(\tau) \to \mathbb{C}$ defined by $\tau'[u] := \tau(u, u)$, $u \in \mathcal{D}(\tau)$ is called the *quadratic form* associated with the form τ. Due to the polarization identity

$$\tau(u, v) = \frac{1}{4}(\tau'[u+v] - \tau'[u-v] + \tau'[u+iv] - \tau'[u-iv])$$

the quadratic form τ' determines the sesquilinear form τ uniquely. This is why we shall use the same notation τ for both sesquilinear and the associated quadratic forms unless this leads to confusion.

We say that two forms τ_1 and τ_2 are equal if and only if $\mathcal{D}(\tau_1) = \mathcal{D}(\tau_2)$ and for all $u, v \in \mathcal{D}(\tau_1)$ the equality $\tau_1(u, v) = \tau_2(u, v)$ holds. If $\mathcal{D}(\tau_1) \subset \mathcal{D}(\tau_2)$ and for all $u, v \in \mathcal{D}(\tau_1)$ we have $\tau_1(u, v) = \tau_2(u, v)$, then the form τ_2 is said to be an *extension* of τ_1 and the form τ_1 is called a *restriction* of τ_2.

The *unit* form $1(u, v)$ equals, by definition, the inner product in $\mathcal{X}$. Therefore for every $a \in \mathbb{C}$ and every form τ the form $\tau + a$ is defined by

$$(\tau + a)(u, v) = \tau(u, v) + a\langle u, v\rangle, \quad u, v \in \mathcal{D}(\tau + a) = \mathcal{D}(\tau).$$

A form τ is said to be *symmetric* if

$$\tau(u, v) = \overline{\tau(v, u)}, \quad u, v \in \mathcal{D}(\tau).$$

With every form τ one can associate another form τ^* defined by

$$\tau^*(u, v) = \overline{\tau(v, u)}, \quad u, v \in \mathcal{D}(\tau^*) = \mathcal{D}(\tau).$$

The form τ^* is called the *adjoint* form of τ. Hence, the form τ is symmetric iff $\tau = \tau^*$.

For an arbitrary form τ one can define two new forms, b and s, by setting

$$b := \frac{1}{2}(\tau + \tau^*), \qquad s = \frac{1}{2i}(\tau - \tau^*).$$

The forms b and s are called the *real* and *imaginary parts* of the form τ respectively. Note that these forms are symmetric and

$$\tau = b + is = \operatorname{Re}\tau + i\operatorname{Im}\tau.$$

A symmetric form b is said to be *bounded below* if the set $\{b[u] : u \in \mathcal{D}(b),\ \|u\| = 1\} \subset \mathbb{R}$ is bounded below or, equivalently, there is a number $\gamma \in \mathbb{R}$ such that

$$b[u] \geq \gamma \|u\|^2, \quad u \in \mathcal{D}(b).$$

This will be written briefly as

$$b \geq \gamma.$$

The largest number γ satisfying this property is called the *lower bound* of b. The form b is called *non-negative* if the inequality $b \geq 0$ holds.

Now we return to the case of general (non-symmetric) form τ. The set

$$\Theta(\tau) := \{\tau[u] : u \in \mathcal{D}(\tau),\ \|u\| = 1\}$$

is called the *numerical range* of τ. This is a convex subset of complex plane. The form τ is said to be bounded from the left if there exists a number $\gamma \in \mathbb{R}$ such that

$$\Theta(\tau) \subset \{z \in \mathbb{C} : \operatorname{Re} z \geq \gamma\}.$$

Let $\gamma \in \mathbb{R}$, $0 < \theta < \pi$. We set

$$S_{\theta,\gamma} := \{z \in \mathbb{C} : |arg\,(z-\gamma)| < \theta\}.$$

A form τ is called *sectorial* if one can find numbers $\gamma \in \mathbb{R}$ and $\theta \in (0, \frac{\pi}{2})$ such that $\Theta(\tau) \subset S_{\theta,\gamma}$. This means that

$$b \geq \gamma, \quad \text{and} \quad |s[u]| \leq \tan\theta(b-\gamma)[u], \quad u \in \mathcal{D}(\tau),$$

where b and s are the real and the imaginary parts of τ. The numbers γ and θ are called a *vertex* and a *semiangle* of the form τ respectively.

Let τ be a sectorial form. A sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ is said to be *τ-convergent* (to an element $u \in \mathcal{X}$) if $u_n \in \mathcal{D}(\tau)$, $u_n \to u$ and $\tau[u_n - u_m] \to 0$ as $n, m \to \infty$. Note that u need not belong to $\mathcal{D}(\tau)$. It is obvious that τ-convergence is equivalent to $(\tau + a)$-convergence for any $a \in \mathbb{C}$.

A sectorial form τ is said to be *closed* if τ-convergence to u implies that $u \in \mathcal{D}(\tau)$ and $\tau[u_n - u] \to 0$. Hence, a form τ is closed iff $\tau + a$ is closed for some $a \in \mathbb{C}$.

Let b be a symmetric non-negative form and set

$$(u, v)_b := (b+1)(u, v) = b(u, v) + (u, v), \quad u, v \in \mathcal{D}(b). \tag{2.1}$$

One can regard $(\cdot,\cdot)_b$ as an inner product in $\mathcal{D}(b)$. Then $\mathcal{D}(b)$ can be regarded as a pre-Hilbert space and will be denoted by H_b.

Let τ be a sectorial form. We define a pre-Hilbert space $\mathcal{X}_\tau := \mathcal{X}_{b'}$, where $b' = \operatorname{Re}\tau - \gamma \geq 0$ and γ is a vertex of τ. One can readily see that a sequence $(u_n)_{n\in\mathbb{N}} \in \mathcal{D}(\tau)$ is τ-convergent iff it is a Cauchy sequence in $\mathcal{X}_\tau$. Then the form τ is bounded in $\mathcal{X}_\tau$ and the following statement holds.

Proposition 2.1.1. *A sectorial form τ is closed if and only if the pre-Hilbert space H_τ is complete.*

A sectorial form is called *closable* if it has a closed extension. The smallest closed extension of the form τ is called the closure of τ and is denoted by $\overline{\tau}$.

Let τ be a closed sectorial form. A linear subset $\mathcal{D}_1 \subset \mathcal{D}$ is called a *core* of τ if $\overline{\tau_1} = \tau$, where $\tau_1 = \tau \restriction_{\mathcal{D}_1}$. The following statement holds.

Proposition 2.1.2. *Let τ_1, τ_2 be sectorial forms in $\mathcal{X}$ and let $\tau = \tau_1 + \tau_2$ (with $\mathcal{D}(\tau) = \mathcal{D}(\tau_1) \cap \mathcal{D}(\tau_2)$). Then τ is sectorial. If both τ_1 and τ_2 are closed, so is τ. If both τ_1 and τ_2 are closable, so is τ and*

$$\overline{\tau} \subset \overline{\tau_1} + \overline{\tau_2}.$$

Let τ be a sectorial form in $\mathcal{X}$. A form τ_1, which need not be sectorial, is said to be *τ-bounded* if $\mathcal{D}(\tau_1) \supset \mathcal{D}(\tau)$ and

$$|\tau_1[u]| \leq \alpha|\tau[u]| + c\|u\|^2, \quad u \in \mathcal{D}(\tau), \tag{2.2}$$

where $\alpha \geq 0$ and $c \in \mathbb{R}$. The greatest lower bound for all possible values of α is called the τ-bound of τ_1. Obviously τ-boundedness is equivalent to $(\tau + a)$-boundedness for every $a \in \mathbb{C}$ as well as to b-boundedness, where $b = \operatorname{Re}\tau$. We complete this subsection by formulating the following important result.

Theorem 2.1.3. *Let τ be a sectorial form and let τ_1 be τ-bounded with τ-bound $c < 1$ in (2.2). Then $\tau + \tau_1$ is sectorial. $\tau + \tau_1$ is closed iff τ is closed and closable iff so is τ, with $\mathcal{D}(\overline{\tau + \tau_1}) = \mathcal{D}(\overline{\tau})$.*

2.1.2 Representation Theorems

In this subsection we collect several fundamental results (known as *representation theorems*) that establish the relation between closed sectorial forms and m-sectorial operators in Hilbert spaces. We begin with recalling some notions of the operator theory.

By analogy with forms we say that an operator $\mathcal{A}$ in the Hilbert space $\mathcal{X}$ is sectorial if the set

$$\Theta(\mathcal{A}) := \{\langle \mathcal{A}u, u\rangle : u \in \mathcal{D}(\mathcal{A}), \|u\| = 1\} \subset S_{\theta,\gamma}$$

for some $\gamma \in \mathbb{R}$ and $0 < \theta \leq \frac{\pi}{2}$. If, in addition, $\mathcal{A}$ is closed and $\gamma - 1 - \mathcal{A}$ is invertible, then the operator $\mathcal{A}$ is said to be *m-sectorial.*

Now we are ready to formulate the First Representation Theorem.

Theorem 2.1.4. *Let τ be a densely defined, closed, sectorial sesquilinear form in $\mathcal{X}$. Then there exists an m-sectorial operator $\mathcal{A}$ such that*

(i) $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\tau)$ and

$$\tau(u, v) = (\mathcal{A}u, v) \tag{2.3}$$

for every $u \in \mathcal{D}(\mathcal{A})$ and $v \in \mathcal{D}(\tau)$;

(ii) $\mathcal{D}(\mathcal{A})$ is a core of τ;

(iii) if $u \in \mathcal{D}(\tau)$, $w \in \mathcal{X}$ and

$$\tau(u, v) = (w, v)$$

holds for every v belonging to a core of τ, then $u \in \mathcal{D}(\mathcal{A})$ and $\mathcal{A}u = v$.

The m-sectorial operator $\mathcal{A}$ is uniquely determined by condition (i).

Remark. *For symmetric forms Theorem 2.1.4 was proved by K. Friedrichs.*

Corollary 2.1.5. *Let τ_0 be the form defined by $\tau_0(u, v) = (\mathcal{A}u, v)$ with $\mathcal{D}(\tau_0) = \mathcal{D}(\mathcal{A})$, where $\mathcal{A}$ is the m-sectorial operator associated with the sectorial form τ in Theorem 2.1.4 above. Then $\tau = \overline{\tau_0}$.*

The Second Representation Theorem holds for symmetric forms only.

Theorem 2.1.6. *Let b be a densely defined, closed symmetric form, $b \geq 0$, and let $\mathcal{B} = \mathcal{A}_b$ be the associated (by the first representation theorem) self-adjoint operator. Then we have $\mathcal{D}(\mathcal{B}^{\frac{1}{2}}) = \mathcal{D}(b)$ and*

$$b(u, v) = (\mathcal{B}^{\frac{1}{2}}u, \mathcal{B}^{\frac{1}{2}}v), \quad u, v \in \mathcal{D}(b).$$

However, Theorem 2.1.6 admits the following generalisation.

Theorem 2.1.7. *Let $\mathcal{B}$ be a m-sectorial operator with a vertex 0 and semi-angle θ. Then $\mathcal{A} := \operatorname{Re}\mathcal{B}$ is non-negative, and there is a bounded self-adjoint operator $\mathcal{G}$ on $\mathcal{X}$ such that $\|\mathcal{G}\| \leq \tan\theta$ and*

$$\mathcal{B} = \mathcal{A}^{\frac{1}{2}}(\mathrm{Id} + i\mathcal{G})\mathcal{A}^{\frac{1}{2}}.$$

2.1.3 Convergence Theorems

In this subsection we state two results which show the relation between convergence of sectorial forms and the strong resolvent convergence of the associated m-sectorial operators. These statements will be of great importance in our further considerations.

Let b_1 and b_2 be symmetric forms in $\mathcal{X}$. We say that $b_1 \leq b_2$ if $\mathcal{D}(b_1) \supset \mathcal{D}(b_2)$ and for all $u \in \mathcal{D}(b_2)$ the inequality $b_1[u] \leq b_2[u]$ holds.

The first theorem, due to B. Simon ([63, Th. S.14]), is valid for symmetric forms.

Theorem 2.1.8. *Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of closed positive quadratic forms satisfying $0 \le b_1 \le \cdots \le b_n \le \cdots$. Suppose that*

$$\mathcal{D}(b) := \left\{u \in X \mid \sup_n b_n[u] < \infty\right\}$$

is dense in $\mathcal{X}$. Then the quadratic form

$$b[u] := \lim_n b_n[u] = \sup_n b_n[u]$$

with domain $\mathcal{D}(b)$ is closed. Moreover, if $\mathcal{A}_n$, $\mathcal{A}$, $n \in \mathbb{N}$ are the operators associated with the forms b_n, b, $n \in \mathbb{N}$, then $\mathcal{A}_n \to \mathcal{A}$ in the strong resolvent sense, i.e. $(\lambda - \mathcal{A}_n)^{-1}x \to (\lambda - \mathcal{A})^{-1}x$ in $\mathcal{X}$ for all $x \in \mathcal{X}$ and $\lambda \in \cap_{n\ge1}\rho(\mathcal{A}_n) \cap \rho(\mathcal{A})$, where $\rho(\mathcal{B})$ stands for the resolvent set of an operator $\mathcal{B}$.

In the next statement (see [72, Lemma 4.7]) the forms are not assumed to be symmetric.

Lemma 2.1.9. *Let τ_n, $n \in \mathbb{N} \cup \{0\}$ be closed sectorial forms and $\mathcal{A}_n$, $n \in \mathbb{N} \cup \{0\}$ the associated m-sectorial operators. We assume that there exist a closed symmetric form $\mathrm{b} \ge 1$ and constants $c \ge 1$ and $\omega \in \mathbb{R}$, such that*

$$c^{-1}\,\mathrm{b} \le \mathrm{Re}\,\tau_n + \omega \le c\,\mathrm{b}$$

and

$$\sup_{\mathrm{b}[v]\le 1} \left|(\tau_0 - \tau_n)(u, v)\right| \to 0 \quad as \quad n \to \infty$$

for all $u \in \mathcal{D}(\mathrm{b})$. Then $\mathcal{A}_n \to \mathcal{A}_0$ in the strong resolvent sense.

2.2 Strongly Continuous Semigroups

Here we collect some basic notions and results of the theory of strongly continuous semigroups of operators on Banach spaces.

Let X be a complex Banach space with norm $\|\cdot\|$. By $L(X)$ we denote the Banach algebra of all bounded linear operators on X. The operator norm in $L(X)$ will be denoted by the same symbol as the norm in X.

2.2.1 Basic Notions and Properties

Definition 2.2.1. *A family $\{T(t),\, t \geq 0\} \subset L(X)$ is called a (one-parameter) semigroup on X if it satisfies the following properties:*

(i) $T(0) = \mathrm{Id}$;

(ii) $T(t+s) = T(t)T(s)$, *for any* $t, s \geq 0$.

A semigroup $T(t)$, $t \geq 0$, is called strongly continuous (or a C_0-semigroup) if

(iii) $\lim_{t\to 0} T(t)x = x$ *for every* $x \in X$.

The generator $\mathcal{A}$ of a semigroup $T(t)$ is the operator in X defined by

$$\mathcal{A}x = \lim_{t\to 0} \frac{T(t)x - x}{t},$$

with the domain $\mathcal{D}(\mathcal{A})$ consisting of all those $x \in X$ for which such a limit exists.

In Remark 2.2.2 below we collect some basic properties of C_0-semigroups.

Remark 2.2.2. *Let $T(t)$ be a C_0-semigroup on X with the generator $\mathcal{A}$.*

(i) $\mathcal{A}$ is a closed operator and $\mathcal{D}(\mathcal{A})$ is dense in X, (see e.g. [62, Cor. 1.2.5]).

(ii) For $x_0 \in \mathcal{D}(\mathcal{A})$ set $u(t) := T(t)x_0$, $t \geq 0$. Then u is continuously differentiable, $u(t) \in \mathcal{D}(\mathcal{A})$ and

$$u'(t) = \mathcal{A}u(t), \quad t \geq 0,$$

(see e.g. [62, Th. 1.2.4]).

(iii) There exist constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|T(t)\| \leq Me^{\omega t} \tag{2.4}$$

for all $t \geq 0$, ([62, Th. 1.2.2]). The set

$$R_\omega := \{\lambda \in \mathbb{C} |\, \mathrm{Re}\,\lambda > \omega\} \subset \rho(\mathcal{A})$$

and

$$(\lambda - \mathcal{A})^{-1}x = \int_0^\infty e^{-\lambda t}T(t)x dt$$

for all $\lambda \in R_\omega$, ([62, Rem. 1.5.4]).

(iv) *For $x \in \mathcal{D}(\mathcal{A})$ and $t \geq 0$ the following equality holds:*

$$\mathcal{A}T(t)x = T(t)\mathcal{A}x,$$

([62, Th. 1.2.4]). Combined with (iii) this implies that

$$(\lambda - \mathcal{A})^{-1}\mathcal{A}x = \mathcal{A}(\lambda - \mathcal{A})^{-1}x$$

and

$$(\lambda - \mathcal{A})^{-1}T(t)x = T(t)(\lambda - \mathcal{A})^{-1}x.$$

(v) *If c is a constant then the operator $\mathcal{A} + c\,(= \mathcal{A} + c\,\mathrm{Id})$ generates the C_0-semigroup $T(t)e^{ct}$, $t \geq 0$.*

Let us consider the abstract Cauchy problem

$$\begin{cases} u'(t) = \mathcal{A}u(t), \ t > 0, \\ u(0) = x_0, \end{cases} \tag{2.5}$$

in X with a linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset X :\to X$ and $x_0 \in X$. Property (ii) implies that if $\mathcal{A}$ is the generator of a C_0-semigroup $T(t)$ and $x_0 \in \mathcal{D}(\mathcal{A})$ then the function $T(\cdot)x_0$ delivers a continuously differentiable solution to (2.5). Actually, even a stronger statement holds.

Proposition 2.2.3. *([62, Th.4.1.3]). Let $\mathcal{A}$ be a densely defined linear operator with non-empty resolvent set. Cauchy problem* (2.5) *has the unique solution $u \in C^1(\mathbb{R}_+, X)$ for all $x \in \mathcal{D}(\mathcal{A})$, iff $\mathcal{A}$ is the generator of a C_0-semigroup $T(t)$. The solution is given by $u(t) = T(t)x$, $t \geq 0$.*

Definition 2.2.4. *If estimate* (2.4) *holds with $\omega = 0$ the semigroup T is said to be bounded; if $M = 1$ in* (2.4)*, then the semigroup T is called quasi-contractive; the semigroup T is said to be contractive (or a semigroup of contractions) if both $\omega = 0$ and $M = 1$ in* (2.4).

It is natural to ask under what conditions a linear operator in X is the generator of a C_0-semigroup. The following theorem, which is due to E. Hille and K. Yosida, provides a characterisation of generators of C_0-semigroups.

Theorem 2.2.5. *([29, Th. I.2.6]). Let $\mathcal{A}$ be a linear operator in the Banach space X. Then the following conditions are equivalent:*

(i) $\mathcal{A}$ is the generator of a C_0-semigroup of contractions on X;

(ii) $\mathcal{A}$ is closed, densely defined and for every $\lambda > 0$ we have $\lambda \in \rho(\mathcal{A})$ and

$$\|\lambda(\lambda - \mathcal{A})^{-1}\| \leq 1;$$

(iii) $\mathcal{A}$ is closed, densely defined and for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$ one has $\lambda \in \rho(\mathcal{A})$ and

$$\|(\lambda - \mathcal{A})^{-1}\| \leq \frac{1}{\operatorname{Re}\lambda}.$$

In many cases Theorem 2.2.5 is difficult to apply since the required estimate on the resolvent may not be easy to verify. Another criterion for an operator to generate a C_0-semigroup is given by a theorem due to G. Lumer and R. Philips. In order to formulate this result we need some new notions.

Definition 2.2.6. *([29, Def. I.3.1]). Let X be a Banach space and X^* its dual. For $x \in X$ and $\phi \in X^*$ set $\langle x, \phi\rangle := \phi(f)$. The mapping $\mathcal{J} : X \to P(X^*)$ defined by*

$$\mathcal{J}(x) := \{\phi \in X^* : |\phi|_*^2 = |x|^2 = \langle x, \phi\rangle\},\ x \in X,$$

is called the duality mapping of X. Due to the Hahn-Banach theorem the set $\mathcal{J}(x)$ is non-empty. Let J be a section of $\mathcal{J}$, i.e. $J : X \to X^*$ and $J(x) \in \mathcal{J}(x)$ for every $x \in X$. The function J is called the duality section. An operator $\mathcal{B}$ in X is called accretive w.r.t. the duality section J if $\operatorname{Re}\langle \mathcal{B}x, J(x)\rangle \geq 0$ for every $x \in \mathcal{D}(\mathcal{B})$. An operator is called accretive if it is accretive w.r.t. some duality section. An accretive operator $\mathcal{B}$ is called m-accretive (or maximal accretive) if $\mathcal{B}$ is closed and $\rho(\mathcal{B}) \cap (-\infty, 0) \neq \emptyset$.

Remark. *The last condition implies that $\mathcal{B}$ has no proper accretive extensions. If X is a Hilbert space, then these assertions are equivalent (see [29, I.3]).*

Below we present some important examples of duality mappings in some specific Banach spaces.

Example 2.2.7. *If $\mathcal{X}$ is a Hilbert space identified with its dual, then $\mathcal{J}$ is the identity mapping (i.e. $\mathcal{J}(x) = \{x\}$) and $\langle\cdot,\cdot\rangle$ is identified with the inner product in X. Next we consider an example when X is not a Hilbert space. Let $(M, \mathcal{M}, \mu)$ be a measure space and $1 < p < \infty$. Let $X = L^p(\mu) \equiv L^p(M, \mathcal{M}, \mu)$ and $\|\cdot\|_p$*

stand for the norm in $L^p(\mu)$. Then $\mathcal{J}(0) = \{0\}$, and for $g \in L^p(\mu)$ such that $g \neq 0$ we have

$$\mathcal{J}(g) = \|g\|_p^{2-p}|g|^{p-1}\mathrm{sgn}\, g \in L^{p'}(\mu),$$

where $p' = (p-1)^{-1}p$, and $\mathrm{sgn}\, g := g/|g|$ if $g \neq 0$ and $\mathrm{sgn}\, g := 0$ otherwise. We introduce the following notation. For measurable f, g, such that $fg \in L^1$, let $\langle f,\, g\rangle := \int_M f(y)\overline{g(y)}d\mu(y)$. Hence an operator $\mathcal{B}$ is accretive in $L^p(\mu)$ if

$$[\mathcal{B}f, f]_p \geq 0 \quad \textit{for all} \quad f \in \mathcal{D}(\mathcal{B}),$$

where the semi-scalar product $[\cdot,\cdot]_p$ in $L^p(\mu)$ is defined as follows:

$$[f, g]_p := \langle f,\, |g|^{p-1}\mathrm{sgn}\, g\rangle \|g\|_p^{2-p}, \quad f, g \in L^p(\mu)$$

Now we are ready to formulate the celebrated Lumer-Philips theorem. This statement proves to be of great importance in our further research.

Theorem 2.2.8. *([29, Th. I.3.3]). An operator $\mathcal{A}$ is the generator of a C_0-semigroup of contractions if and only if $-\mathcal{A}$ is densely defined and m-accretive.*

The following fundamental result, known as the Trotter-Kato-Neveu theorem, reveals the intimate relation between the strong convergence of C_0-semigroups and the convergence of their generators in the strong resolvent sense.

Theorem 2.2.9. *([29, Th. I.7.3]). Let $\mathcal{A}$, $\mathcal{A}_n$, $n \in \mathbb{N}$ be a sequence of the generators of C_0-semigroups $T(t)$, $T_n(t)$, $t \geq 0$, $n \in \mathbb{N}$ respectively. We assume that for all $n \in \mathbb{N}$ and $t \geq 0$ there are constants $M \geq 1$ and $\omega \in \mathbb{R}$, independent of n, such that $\|T_n(t)\| \leq M\exp(\omega t)$ and $\|T(t)\| \leq M\exp(\omega t)$.*

(i) If $T_n(t)x \to T(t)x$ for all $t \geq 0$ and $x \in X$, then

$$(\lambda - \mathcal{A}_n)^{-1}x \to (\lambda - \mathcal{A})^{-1}x \textit{ for all } x \in X,$$

and the convergence is uniform in λ from all compact subsets of (ω, ∞);

(ii) If $(\lambda - \mathcal{A}_n)^{-1}x \to (\lambda - \mathcal{A})^{-1}x$ for all $x \in X$ and $\lambda > \omega$, then

$$T_n(t)x \to T(t)x \textit{ for all } x \in X,$$

and the convergence is uniform in t from all compact subsets of $\mathbb{R}_+$.

2.2.2 Analytic Semigroups

Let $0 < \theta \leq \pi$. We use the following notation:

$$S_\theta := S_{\theta,0} = \{z \in \mathbb{C} : z \neq 0, \, |arg\, z| < \theta\}.$$

Definition 2.2.10. *Let $0 < \alpha \leq \pi/2$ and $M \geq 1$. A family of operators $T(z)$, $z \in S_\alpha \cup \{0\}$ is called an analytic semigroup of angle α if*

(i) $T(z_1)T(z_2) = T(z_1 + z_2)$ for all $z_1, z_2 \in S_\alpha$, $T(0) = \mathrm{Id}$;

(ii) for all $x \in X$ and $\varphi \in X^*$ the function $\langle T(\cdot)x, \varphi\rangle$ is analytic in S_α;

(iii) $\lim\limits_{t\to 0,\, t\in S_{\alpha-\varepsilon}} T(t)x = x$ for all $x \in X$ and $\varepsilon \in (0, \alpha)$.

If, in addition,

(iv) for every $\varepsilon \in (0, \alpha)$ there is a number $\omega_\varepsilon \in \mathbb{R}$ such that for all $z \in S_{\alpha-\varepsilon}$ we have $\|T(z)\| \leq e^{\omega_\varepsilon |z|}$, we say that $T(z)$, $z \in S_\alpha \cup \{0\}$, is an analytic semigroup of quasi-contractions.

The next theorem, due to M. Solomyak and K. Yosida, characterises generators of analytic semigroups on X.

Theorem 2.2.11. *A closed densely defined operator $\mathcal{A}$ generates an analytic semigroup if and only if there exist constants $M > 0$ and $\omega \in \mathbb{R}$ such that $R_\omega \subset \rho(\mathcal{A})$ and $\|(\mathcal{A} - \lambda)^{-1}\| \leq \frac{M}{1+|\lambda|}$ for all $\lambda \in R_\omega$ (recall that the set R_ω was defined in Remark 2.2.2, (iii)).*

The following result shows a relation between well-posedness of the abstract Cauchy problems and analytic semigroups.

Proposition 2.2.12. *If $\mathcal{A}$ is the generator of an analytic semigroup on X then the Cauchy problem* (2.5) *has a unique solution for all $x_0 \in X$. The solution is given by $u(t) = \exp(t\mathcal{A})x_0$, $t \geq 0$.*

Before formulating a statement which characterises generators of analytic quasi-contractive semigroups, we need to extend the notion of m-sectorial operator to the case of Banach spaces.

Definition 2.2.13. *Let $\mathcal{B}$ be a linear operator in X. The set*

$$\Theta(\mathcal{B}) := \{\langle \mathcal{B}x,\, \phi\rangle : x \in \mathcal{D}(\mathcal{B}),\ \|x\| = 1,\ \phi \in \mathcal{J}(x)\}$$

is called the numerical range of the operator $\mathcal{B}$. The operator $\mathcal{B}$ is said to be sectorial if there are numbers $\theta \in (0, \pi/2)$ and $\omega \in \mathbb{R}$ such that $\Theta(\mathcal{B} - \omega) \subset S_\theta$. We say that the operator $\mathcal{B}$ is m-sectorial if $\mathcal{B}$ is sectorial and $\mathcal{B} - \omega$ is m-accretive.

Remark. *We note that Definition 2.2.13 is consistent with the definition of m-sectorial operator, given in subsection 2.1.2.*

Proposition 2.2.14. *The following two statements are equivalent:*

(i) the operator $\mathcal{A}$ generates an analytic semigroup of quasi-contractions on X;

(ii) the operator $-\mathcal{A}$ is m-sectorial.

Let $\mathcal{X}$ be a Hilbert space and τ a closed densely defined sectorial form in $\mathcal{X}$. Then by Theorem 2.1.4 there is a unique m-sectorial operator $\mathcal{B}$ associated with the form τ. Proposition 2.2.14 implies that $-\mathcal{B}$ is the generator of a quasi-contractive analytic semigroup T. The semigroup T is said to be associated with the form τ. The following theorem provides a converse statement.

Theorem 2.2.15. *Let T be a quasi-contractive analytic semigroup on the Hilbert space $\mathcal{X}$. For $t > 0$ set $\tau_t(u, v) := t^{-1}\langle u - T(t)u,\, v\rangle$, $u, v \in \mathcal{X}$. Then T is associated with a densely defined sectorial form τ and the following assertions hold:*

(i) $u \in \mathcal{D}(\tau)$ iff $\sup\limits_{t>0} \tau_t(u, u) < \infty$;

(ii) for all $u, v \in \mathcal{D}(\tau)$ we have $\lim\limits_{t\to 0} \tau_t(u, v) = \tau(u, v)$.

(iii) $\lim\limits_{t\to 0} (\tau + 1)[u - T(t)u] = 0$ for all $u \in \mathcal{D}(\tau)$.

2.2.3 Sub-Markovian Semigroups and Dirichlet Forms

Here we give a brief overview of the theory of sub-Markovian semigroups. Most of the statements with the proofs can be found in [27] or [58].

Let $(M, \mathcal{M}, \mu)$ be a σ-finite measure space. As before $L^p(\mu) := L^p(M, \mu)$, $p \geq 1$. We shall use the following notation: $\mathrm{Re}\, L^p(\mu) := \{f \in L^p(\mu) : f \text{ is real}\}$, $L^p_+(\mu) := \{f \in L^p(\mu) : f \geq 0 \ a.e.\}$. For a measurable real-valued function u we set $u^+ := \sup\{u, 0\}$ and $u^- := \sup\{-u, 0\}$. In particular, $u = u^+ - u^-$ for all $u \in \mathrm{Re}\, L^p(\mu)$.

Definition 2.2.16. *A contractive C_0-semigroup T^p on $L^p(\mu)$ is said to be real if $T^p \operatorname{Re} L^p(\mu) \subset \operatorname{Re} L^p(\mu)$. A real semigroup T^p is called positive (or positivity preserving) if $T^p L^p_+(\mu) \subset L^p_+(\mu)$, $t \geq 0$. It is said to be $L^\infty(\mu)$-contractive if $\|T^p(t)f\|_\infty \leq \|f\|_\infty$ for all $f \in L^p(\mu) \cap L^\infty(\mu)$. We shall say that T^p is sub-Markovian if it is positive and $L^\infty(\mu)$-contractive.*

First we concentrate on the case $p = 2$. The following statement holds.

Proposition 2.2.17. *Let T be a semigroup on $L^2(\mu)$ associated with a closed sectorial form τ. Let $-\mathcal{A}$ stand for the generator of T. The following assertionss are equivalent:*

(i) the semigroup T is real;

(ii) if $u \in \mathcal{D}(\mathcal{A})$ then $\overline{u} \in \mathcal{D}(\mathcal{A})$ and $\mathcal{A}\overline{u} = \overline{\mathcal{A}u}$;

(iii) if $u \in \mathcal{D}(\tau)$ then $\overline{u} \in \mathcal{D}(\tau)$ and $\tau(u, v) = \overline{\tau(\overline{u}, \overline{v})}$ for all $u, v \in \mathcal{D}(\tau)$;

(iv) if $u \in \mathcal{D}(\tau)$ then $\operatorname{Re} u \in \mathcal{D}(\tau)$ and $\tau(u, v) \in \mathbb{R}$ for all $u, v \in \mathcal{D}(\tau) \cap \operatorname{Re} L^2(\mu)$.

A sesquilinear form is called *real* if it satisfies condition (iii) of the previous statement. A real form τ is said to satisfy the *first Beurling-Deny* condition if for all $u \in \mathcal{D}(\tau) \cap \operatorname{Re} L^2(\mu)$ one has $u^+ \in \mathcal{D}(\tau)$ and $\tau(u^+, u^-) \leq 0$. One can readily see that if τ satisfies the first Beurling-Deny condition, then so does $\operatorname{Re} \tau$. Next we characterise positive semigroups in terms of the corresponding sesquilinear forms.

Proposition 2.2.18. *Let T be a real semigroup on $L^2(\mu)$ associated with a closed sectorial form τ. Then the following statements are equivalent.*

(i) the semigroup T is positive;

(ii) the form τ satisfies the first Beurling-Deny condition.

Now we formulate a result which concerns the characterisation of $L^\infty(\mu)$-contractivity of semigroups.

Proposition 2.2.19. *Let T be a C_0-semigroup on $L^2(\mu)$ associated with a closed sectorial form τ. Then the following statements are equivalent:*

(i) the semigroup T is $L^\infty(\mu)$-contractive;

(ii) if $u \in \mathcal{D}(\tau)$*, then* $(|u|-1)^+ \operatorname{sgn} u \in \mathcal{D}(\tau)$ *and*

$$\operatorname{Re} \tau(u,\, (|u|-1)^+ \operatorname{sgn} u) \geq 0.$$

Next we are going to introduce the notion of *Dirichlet form* and reveal its intimate relation to sub-Markovian semigroups. We start with the following useful characterisation result.

Proposition 2.2.20. *Let* T *be a* C_0*-semigroup on* $L^2(\mu)$ *associated with a closed densely defined sectorial form* $\tau \geq 0$*. Then the following statements are equivalent.*

(i) The semigroup T *is sub-Markovian.*

(ii) The form τ *is real and for all real* $u \in \mathcal{D}(\tau)$ *we have* $u^+ \wedge 1 \in \mathcal{D}(\tau)$ *and* $\tau\big(u^+ \wedge 1,\, u - (u^+ \wedge 1)\big) \geq 0$.

(iii) The form τ *is real and for all real* $u \in \mathcal{D}(\tau)$ *we have* $u^+ \wedge 1 \in \mathcal{D}(\tau)$ *and* $\tau\big(u + (u^+ \wedge 1),\, u - (u^+ \wedge 1)\big) \geq 0$.

Below we give the definition of Dirichlet form.

Definition 2.2.21. *A sesquilinear form* τ *in* $L^2(\mu)$ *is called a Dirichlet form if* τ *is densely defined sectorial and closed,* $\operatorname{Re} \tau \geq 0$ *and for all real* $u \in \mathcal{D}(\tau)$ *we have* $u^+ \wedge 1 \in \mathcal{D}(\tau)$ *and*

$$\tau(u \pm u^+ \wedge 1,\, u \mp u^+ \wedge 1) \geq 0. \tag{2.6}$$

The next proposition states that to ensure that a given form is a Dirichlet form it suffices to verify condition (2.6) on a dense subset of $\mathcal{D}(\tau)$.

Proposition 2.2.22. *Let* τ *be a densely defined closed sectorial form and* $\mathcal{D}_0 \subset \mathcal{D}(\tau)$ *be a core of* τ*. Then* τ *is a Dirichlet form if* (2.6) *holds for all* $u \in \mathcal{D}_0$.

Recall that a function $g : \mathbb{C} \to \mathbb{C}$ is called a *normal contraction* if $g(0) = 0$ and $|g(z_1) - g(z_2)| \leq |z_1 - z_2|$ for all $z_1, z_2 \in \mathbb{C}$. The following important property of Dirichlet forms can be found in, e.g., [65, Th.XIII.51]. Assume that τ is a Dirichlet form. Then for all $u \in \mathcal{D}(\tau)$ we have $g \circ u \in \mathcal{D}(\tau)$.

The definition of Dirichlet form and interpolation yield the following result.

Proposition 2.2.23. *Let* T *be a* C_0*-semigroup on* $L^2(\mu)$ *associated with a closed densely defined sectorial form* τ*. Then the following assertions are equivalent.*

(i) The form τ is a Dirichlet form.

(ii) The semigroup T is positive and $L^p(\mu)$-contractive for all $1 \le p \le \infty$.

Proposition 2.2.23 implies that a Dirichlet form τ gives rise to a family T^p of C_0-semigroups of contractions on $L^p(\mu)$, $1 \le p < \infty$. These semigroups are consistent in the sense that $T^{p_1} = T^{p_2}$ on $L^{p_1}(\mu) \cap L^{p_2}(\mu)$ for all $1 \le p_1 < p_2 < \infty$. Furthermore, the semigroups T^p, $1 \le p < \infty$, are sub-Markovian.

Indeed, let T be a C_0-semigroup associated with the Dirichlet form τ. Proposition 2.2.23 implies that $\|T(t)f\|_p \le \|f\|_p$ for all $f \in L^2(\mu) \cap L^p(\mu)$ and $t \ge 0$. Hence, for every $1 \le p < \infty$ we can define the semigroup T^p on $L^p(\mu)$ by

$$T^p := (T \restriction_{L^2(\mu)\cap L^p(\mu)})^{\sim}_{L^p(\mu)\to L^p(\mu)}.$$

One can readily see that T^p is strongly continuous, positive and L^∞-contractive for every $1 \le p < \infty$.

The conditions introduced in Propositions 2.2.18 and 2.2.19 can be extended to the case of the space $L^p(\mu)$ with $p \ne 2$. The following result holds.

Theorem 2.2.24. *Let T^p be a C_0-semigroup on $L^p(\mu)$ and $-\mathcal{A}_p$ stand for its generator. Then the following statements are equivalent.*

(i) The semigroup T^p is sub-Markovian.

(ii) For all $f \in \mathcal{D}(\mathcal{A}_p) \cap \operatorname{Re} L^p(\mu)$ we have $[\mathcal{A}_p f, f^+]_p \ge 0$, and for all $f \in \mathcal{D}(\mathcal{A}_p)$ the inequality

$$\operatorname{Re}[\mathcal{A}_p f, (|f|-1)^+ \operatorname{sgn} f]_p \ge 0$$

holds, where $[\cdot, \cdot]_p$ is the semi-scalar product in $L^p(\mu)$, introduced in Example 2.2.7.

Now we assume that a Dirichlet form $\mathcal{E}$ is symmetric. Let T^p be a sub-Markovian semigroup on $L^p(\mu)$ associated with the form $\mathcal{E}$. In this case the generator $-\mathcal{A}_p$ of T^p is said to be symmetric. In Theorem 2.2.25 we collect some remarkable properties of symmetric Markov generators.

Set $\kappa(p) := \sup_{s\in(0,1)} \big((1+s^{\frac{1}{p}})(1+s^{\frac{1}{p'}})(1+s^{\frac{1}{2}})^{-2}\big)$.

Theorem 2.2.25. *Let $1 < p < \infty$. Let $f \in \mathcal{D}(\mathcal{A}_p)$. Then $f_p := f|f|^{\frac{p-2}{2}} \in \mathcal{D}(\tau_\mathcal{A})$ and*

$$4\frac{p-1}{p^2}\tau_\mathcal{A}[f_p] \le \operatorname{Re}\langle \mathcal{A}_p f, |f|^{p-1}\operatorname{sgn} f\rangle \le \kappa(p)\tau_\mathcal{A}[f_p],$$

$$\left|\mathrm{Im}\left\langle \mathcal{A}_p f,\, |f|^{p-1}\mathrm{sgn}\, f\right\rangle\right| \le \frac{|p-2|}{2\sqrt{p-1}}\,\mathrm{Re}\left\langle \mathcal{A}_p f,\, |f|^{p-1}\mathrm{sgn}\, f\right\rangle,$$

where $\langle\cdot,\,\cdot\rangle$ *stands for the inner product in* $L^2(\mu)$. *If, in addition,* $f \in \mathcal{D}(\mathcal{A}_p) \cap L^p_+$ *then* $f^{\frac{p}{2}} \in \mathcal{D}(\tau_{\mathcal{A}})$ *and*

$$4\frac{p-1}{p^2}\tau_{\mathcal{A}}[f^{\frac{p}{2}}] \le \left\langle \mathcal{A}_p f,\, f^{p-1}\right\rangle \le \kappa(p)\tau_{\mathcal{A}}[f^{\frac{p}{2}}].$$

2.3 C_0-semigroups Associated with Elliptic Differential Expressions

We are concerned with the study of second order elliptic operators which generate C_0-semigroups. In this section we discuss several ways to associate a C_0-semigroup with a formal second order differential expression.

Let $d \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$. Let $1 \le p < \infty$ and $L^p \equiv L^p(\Omega,\, \mathcal{B}(\Omega),\, \mu)$. We restrict ourselves to the case $p < \infty$ since the generators of semigroups on L^∞ ususally don't have dense domains in L^∞. These semigroups are therefore not strongly continuous. We begin with introducing some notation. Let $a : \Omega \to \mathbb{R}^d \otimes \mathbb{R}^d$, $b : \Omega \to \mathbb{R}^d$ and $q : \Omega \to \mathbb{C}$ be measurable. By $\beta^\mu := (\beta^\mu_1, \ldots, \beta^\mu_d)$ we denote the logarithmic derivative of the measure μ.

We introduce the differential expression

$$\Lambda := -(\nabla + \beta^\mu)\cdot a \cdot \nabla + b \cdot \nabla + q,$$

i.e.

$$\begin{aligned}(\Lambda u)(x) = &-\sum_{k,j=1}^{d}\left(\frac{\partial}{\partial x_j} + \beta^\mu_j(x)\right)\left(a_{jk}(x)\frac{\partial u}{\partial x_k}(x)\right)\\ &+\sum_{k=1}^{d}\left(b_k(x)\frac{\partial u}{\partial x_k}(x)\right) + q(x)u(x).\end{aligned} \tag{2.7}$$

2.3.1 The Form Method. Miyadera Theorem

It is the form method that is frequently employed in order to define elliptic operators in divergence form with measurable coefficients. The sesquilinear form associated with the differential expression Λ is

$$\tau_\Lambda(u, v) = \langle \nabla u \cdot a \cdot \nabla v\rangle + \langle b \cdot \nabla u,\, v\rangle + \langle qu,\, v\rangle \tag{2.8}$$

on a suitable domain responding to boundary conditions.

The corresponding C_0-semigroup can be constructed as follows. We assume the form τ_Λ is densely defined and sectorial. Then by Theorem 2.2.14 the closure of τ_Λ gives rise to an analytic quasi-contractive semigroup T on $L^2(\mu)$. Let $p \neq 2$. We assume that for all $f \in L^2(\mu) \cap L^p(\mu)$ the estimate $\|T(t)f\|_p \leq \exp(k_p t)\|f\|_p$, $t \geq 0$, holds. Then one can construct a quasi-contractive C_0-semigroup T^p on $L^p(\mu)$, setting

$$T^p(t) := \left(T(t) \restriction_{L^2(\mu)\cap L^p(\mu)}\right)^{\sim}_{L^p(\mu)}, \ t \geq 0.$$

It is clear that the constructed semigroup T^p is consistent to T in the sense that $T^p(t) \restriction_{L^2(\mu)\cap L^p(\mu)} = T(t) \restriction_{L^2(\mu)\cap L^p(\mu)}$, $t \geq 0$.

This method is applicable if, for example, the (closure of the) form τ_Λ is a Dirichlet form. Then we obtain a family T^p, $1 \leq p < \infty$, of contractive C_0-semigroups (see Proposition 2.2.23 and the discussion afterwards). The same technique allows one to obtain a C_0-semigroup on $L^p(\mu)$ when p only belongs to an interval in $[1, \infty)$, containing 2 (see [39]).

Next we discuss another method which is often used in the perturbation theory of C_0-semigroups on $L^1(\mu)$. First we formulate an abstract result known as the Miyadera Perturbation theorem (see [76]; see also [23, Cor. 3.16]).

Theorem 2.3.1. *Let X be a Banach space. We assume that $-\mathcal{A}$ is the generator of a C_0-semigroup on X, i.e. there are numbers $M \geq 1$ and $\omega \geq 0$ such that $\|\exp(-t\mathcal{A})\| \leq M\exp(\omega t)$ (see (2.4)). Let $\mathcal{B}$ be a linear operator in X and suppose that $\mathcal{B}$ is $\mathcal{A}$-bounded. We also assume that there exist numbers $t_0 > 0$ and $\alpha = \alpha(t_0) \in [0, 1)$ such that*

$$\int_0^{t_0} \exp(-\omega s)\|\mathcal{B}\exp(-s\mathcal{A})x\|ds \leq \alpha\|x\|, \quad x \in \mathcal{D}(\mathcal{A}).$$

Then

(i) the operator $-\mathcal{A} - \mathcal{B}$ with the domain $\mathcal{D}(\mathcal{A} + \mathcal{B}) = \mathcal{D}(\mathcal{A})$ generates a C_0-semigroup on X;

(ii) for all $x \in \mathcal{D}(\mathcal{A})$ and $t \geq 0$ we have

$$\|\exp(-t(\mathcal{A} + \mathcal{B}))x\| \leq \frac{M}{1-\alpha}\exp\left(\left(\omega + \frac{1}{t_0}\log\frac{M}{1-\alpha}\right)t\right);$$

(iii) the semigroup $\exp\big(-t(\mathcal{A}+\mathcal{B})\big)$ *can be represented as*

$$\exp\big(-t(\mathcal{A}+\mathcal{B})\big) = \sum_{n=0}^{\infty} I_n(t),$$

where $I_0(t) := \exp(-t\mathcal{A})$ *and* $I_{n+1}(t) = \int_0^t I_n(t-s)\mathcal{B}I_0(s)ds.$

In general Theorem 2.3.1 is hard to apply since the required estimate is not very easy to derive. However, the situation changes drastically if we take $X := L^1(\Omega, dx) =: L^1(\Omega)$, where $\Omega \subset \mathbb{R}^d$. We apply Theorem 2.3.1 to construct the generator of a C_0-semigroup on $L^1(\Omega)$, associated with the differential expression Λ in the case $q = \operatorname{Re} q = V$. Let a be a positive matrix with bounded measurable entries. Then the form

$$\tau_a(u,v) := \langle \nabla u \cdot a \cdot \nabla v \rangle, \quad u,v \in \mathcal{D}(\tau_a),$$

is a Dirichlet form. Hence, using the form method we construct a C_0-semigroup U^1 on $L^1(\Omega)$, associated with the form τ_a. By $-\mathcal{A}_1$ we denote the generator of U^1.

Let $\mathcal{B}$ be defined by $\mathcal{B}u = b \cdot \nabla u + Vu$, $u \in \mathcal{D}(\mathcal{B})$, where

$$\mathcal{D}(\mathcal{B}) := \big\{u \in L^1(\Omega) \,|\, |\nabla u| \in L^1_{\mathrm{loc}}(\Omega),\, b \cdot \nabla u,\, Vu \in L^1(\Omega)\big\}.$$

By Theorem 2.3.1, if there are numbers $t_0 > 0$ and $0 \le \alpha < 1$ such that

$$\int_0^{t_0} \big(\|b \cdot \nabla \exp(-s\mathcal{A})f\|_1 + \|V \exp(-s\mathcal{A})f\|_1\big)ds \le \alpha \|f\|_1, \quad f \in \mathcal{D}(\mathcal{A}),$$

then the operator $-H := -\mathcal{A} - \mathcal{B}$ with the domain $\mathcal{D}(H) = \mathcal{D}(\mathcal{A})$ generates a C_0 semigroup on $L^1(\Omega)$. For more details and examples see subsections 2.6.1 and 2.6.2.

2.3.2 Potential Perturbations of Generators

In [47] V. Liskevich and A. Manavi managed to associate a natural C_0-semigroup on $L^p(\mu) := L^p(M, \mathcal{M}, \mu)$, $1 \le p < \infty$, with the formal expression $\mathcal{A}+q$, where $-\mathcal{A}$ is the generator of a C_0-semigroup S and q is a complex-valued measurable function. By "natural" we mean that the generator of the constructed C_0-semigroup extends $-\mathcal{A} - q \restriction_{\mathcal{D}}$, where $\mathcal{D} \subset \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(q)$. The main assumption on S is that it is dominated by some positive semigroup U. Below we describe this approach in more details and give some applications.

Let $U(t)$, $t \geq 0$, be a positive C_0-semigroup on $L^p(\mu)$, $1 \leq p < \infty$. We denote its generator by $\mathcal{B}$. Let $S(t)$, $t \geq 0$ be another C_0-semigroup on $L^p(\mu)$ which is dominated by U, i.e.

$$|S(t)f| \leq U(t)|f|, \; f \in L^p(\mu), t \geq 0.$$

Let $q : M \to \mathbb{C}$ be measurable. By q_n, $n \in \mathbb{N}$, we denote the truncation functions of q, i.e. $q_n := (|q| \wedge n)\operatorname{sgn} q$, where $\operatorname{sgn} q = q|q|^{-1}$ if $q \neq 0$, and $\operatorname{sgn} q = 0$ otherwise. Note that for every $n \in \mathbb{N}$ the potential $q_n \in L^\infty(\mu)$.

Let $V \geq 0$ and measurable. Then $0 \leq U_{V_{n+1}}(t) \leq U_{V_n}(t)$, $n \in \mathbb{N}$, where $U_{V_n}(t)$, $t \geq 0$ is the C_0-semigroup generated by $\mathcal{B} - V_n$, and there exists the limit

$$U_V(t) = s\text{-}\lim_n U_{V_n}(t) \tag{2.9}$$

for all $t \geq 0$. The potential V is called U-admissible if U_V is a C_0-semigroup.

A potential $V \geq 0$ is called U-regular if it is U-admissible and for all $t \geq 0$ we have

$$U(t) = s\text{-}\lim_n U_{V-V_n}(t).$$

The following statement holds.

Proposition 2.3.2. *([47], Prop. 1.19). Let V be U-admissible and $|W|$ be U-regular. We assume that the semigroup S is dominated by a positive semigroup U. Then the limit*

$$(S_{iW})_V(t) = s\text{-}\lim_{n,m} S_{V_n+iW_m}(t) = (S_V)_{iW}(t) =: S_{V+iW}(t)$$

exists for all $t \geq 0$ and S_{V+iW} is a C_0-semigroup.

The following observation is due to J. Voigt ([77, Cor. 2.7]). Let $-\mathcal{A}$ be the generator of a C_0-semigroup S on $L^p(\mu)$. Assume that the limit

$$S_q(t) = s\text{-}\lim_n S_{q_n}(t)$$

exists for all $t \geq 0$ and S_q is a C_0-semigroup. Then its generator $-\mathcal{A}(q)$ extends the operator difference $-\mathcal{A} - q$.

Indeed, the Trotter-Kato-Neveu theorem (see Theorem 2.2.9) implies that $(\lambda + \mathcal{A} + q_n)^{-1}g \to (\lambda + \mathcal{A}(q))^{-1}g$ in $L^p(\mu)$ for all $g \in L^p(\mu)$ and sufficiently large $\lambda > 0$. Let $f \in \mathcal{D}(\mathcal{A} + q) = \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(q)$. We note that for all $f \in \mathcal{D}(\mathcal{A} + q)$ we have

$(\lambda+\mathcal{A}+q_n)f \to (\lambda+\mathcal{A}+q)f$ in $L^p(\mu)$, since since $q_n \to q$ μ-a.e. and $qf \in L^p(\mu)$. Therefore

$$f = \lim_n (\lambda + \mathcal{A} + q_n)^{-1}(\lambda + \mathcal{A} + q_n)f = (\lambda + \mathcal{A}(q))^{-1}(\lambda f + \mathcal{A}f + qf),$$

Thus $f \in \mathcal{D}(\mathcal{A}(q))$ and $\mathcal{A}(q)f = \mathcal{A}f + qf$.

Hence, making use of Proposition 2.3.2 one can construct the generator of a C_0-semigroup, associated with the expression Λ in the case $b = 0$ even if the form τ_Λ is not sectorial.

The following lemma is a simple consequence of the Trotter-Kato-Neveu theorem.

Lemma 2.3.3. *Let $(S_n)_{n\in\mathbb{N}}$, S be C_0-semigroups on a Banach space X with the generators $(\mathcal{A}_n)_{n\in\mathbb{N}}$, $\mathcal{A}$ respectively and let the operator $\mathcal{B}$ be bounded. Assume that $S(t) = s\text{-}\lim_{n\to\infty} S_n(t)$. Then $S_\mathcal{B}(t) = s\text{-}\lim_{n\to\infty} S_{n,\mathcal{B}}(t)$, where $S_\mathcal{B}(t)$ and $S_{n,\mathcal{B}}(t)$ are generated by $\mathcal{A} - \mathcal{B}$ and $\mathcal{A}_n - \mathcal{B}$ respectively.*

Lemma 2.3.4. *Let $-\mathcal{A}_p$ be the generator of a positive semigroup T on $L^p(\mu) =: L^p$. Let $V, W \in L^p(\mu)$, real-valued and $V \geq 0$ (observe that under these assumptions V is T-admissible and $|W|$ is T-regular). We set $q := V + iW$. By $-\mathcal{A}_p(q)$ we denote the generator of the C_0-semigroup $T_q(t) = s\text{-}\lim_n \exp(-t(\mathcal{A}_p + q_n))$ (it follows from Proposition 2.3.2 that T_q is a C_0-semigroup). Then for all $\lambda > 0$ the set $(\lambda + \mathcal{A}_p(q))^{-1}\Big(L^1(\mu) \cap L^\infty(\mu)\Big) \subset \mathcal{D}(\mathcal{A}_p) \cap \mathcal{D}(q)$.*

Proof. Let $f \in L^1(\mu) \cap L^\infty(\mu)$. First we claim that $(\lambda + \mathcal{A}_p(q))^{-1}f \in L^\infty(\mu)$.

Indeed, we have

$$(\lambda + \mathcal{A}_p(q))^{-1}f = \int_0^\infty \exp(-\lambda t)T_q(t)f dt.$$

The Trotter formula and Proposition 2.3.2 imply that

$$\int_0^\infty \exp(-\lambda t)T_q(t)f\, dt = \lim_n \lim_k \int_0^\infty \exp(-\lambda t)\left(T\Big(\frac{t}{k}\Big)\exp\Big(-\frac{tq_n}{k}\Big)\right)^k f\, dt.$$

Next we note that $\left|\exp\Big(-\frac{tq_n}{k}\Big)g\right| \leq g$, for all $g \in L^1(\mu) \cap L^\infty(\mu)$, $g \geq 0$, and $k, n \in \mathbb{N}$, and conclude that

$$|(\lambda + \mathcal{A}_p(q))^{-1}f| \leq (\lambda + \mathcal{A}_p)^{-1}|f|.$$

This proves the claim and therefore $(\lambda + \mathcal{A}_p(q))^{-1}f \in \mathcal{D}(q)$.

The potentials q_n, $n \in \mathbb{N}$, are bounded, therefore the second resolvent identity implies that

$$(\lambda + \mathcal{A}_p + q_n)^{-1}f = (\lambda + \mathcal{A}_p)^{-1}f - (\lambda + \mathcal{A}_p)^{-1}q_n(\lambda + \mathcal{A}_p + q_n)^{-1}f. \tag{2.10}$$

Making use of the Trotter-Kato-Neveu theorem and taking into account that $q_n \to q$ μ-a.e. we conclude that $q_n(\lambda + \mathcal{A}_p + q_n)^{-1}f \to q(\lambda + \mathcal{A}_p(q))^{-1}f$ μ-a.e., and the claim yields $q(\lambda+\mathcal{A}_p(q))^{-1}f \in L^p(\mu)$. Thus by the dominated convergence theorem

$$q_n(\lambda + \mathcal{A}_p + q_n)^{-1}f \to q(\lambda + \mathcal{A}_p(q))^{-1}f \text{ in } L^p(\mu).$$

Passing to the limit in (2.10) we obtain

$$(\lambda + \mathcal{A}_p(q))^{-1}f = (\lambda + \mathcal{A}_p)^{-1}f - (\lambda + \mathcal{A}_p)^{-1}q(\lambda + \mathcal{A}_p(q))^{-1}f.$$

Therefore $(\lambda + \mathcal{A}_p(q))^{-1}f \in \mathcal{D}(\mathcal{A}_p)$. □

Next we present an important example of a positive semigroup (namely the semigroup generated by the Dirichlet operator) and discuss which potentials are admissible and regular w.r.t. this semigroup.

Example 2.3.5. *Let $M = \Omega \subset \mathbb{R}^d$, $d\mu = \rho dx$, where $\rho \in L^1_{\text{loc}}(\Omega, dx)$. We write $\langle f \rangle := \int_\Omega f \rho dx$ for $f \in L^1(\rho) := L^1(\Omega, \rho dx)$, and $\langle f, g \rangle := \langle f\bar{g} \rangle$, provided $fg \in L^1(\rho)$. For $\mathbb{C}^d$-valued functions f_1, g_1 let $\langle f_1, g_1 \rangle := \langle f_1 \cdot g_1 \rangle$. By $\mathcal{L}$ we denote the self-adjoint operator in $L^2(\rho)$ associated (by Theorem 2.1.4) with the closure of the form*

$$\mathcal{E}(u, v) = \langle \nabla u, \nabla v \rangle, \quad u, v \in C_0^1(\Omega).$$

One can check that the closure of $\mathcal{E}$ is a Dirichlet form. Thus, using Proposition 2.2.23, we obtain a family U^p, $1 \le p < \infty$, of consistent sub-Markovian semigroups in $L^p(\rho)$. By $-\mathcal{L}_p$ we denote the generator of U^p. The operator $\mathcal{L}_p$ is called the Dirichlet operator in $L^p(\rho)$. The semigroups U^p, $1 \le p < \infty$, are consistent, i.e. $U^{p_1} \upharpoonright_{L^{p_1}(\rho) \cap L^{p_2}(\rho)} = U^{p_2} \upharpoonright_{L^{p_1}(\rho) \cap L^{p_2}(\rho)}$ for all $1 \le p_1 < p_2 < \infty$. By $\beta = (\beta_1, \ldots, \beta_d)$ we denote the logarithmic derivative of the measure $\rho\, dx$, i.e. we assume that the following integration by parts formula holds:

$$\left\langle \frac{\partial f}{\partial x_k} \right\rangle = -\langle \beta_k f \rangle, \quad 1 \le k \le d, \quad f \in C_0^1(\mathbb{R}^d).$$

We observe that if $\beta \in L^p_{\text{loc}}$, then $\mathcal{L}_p \supset -\Delta - \beta \cdot \nabla \upharpoonright_{C_0^2(\mathbb{R}^d)}$.

Till the end of this subsection we work in the framework of Example 2.3.5. The following proposition was proved in [77] (Prop. 5.8) in the case $\rho \equiv 1$.

Proposition 2.3.6. *Let $V : \Omega \to \mathbb{R}_+$. Then*

(i) the potential V is U^p-admissible for some (all) $1 \le p < \infty$ iff $Q(\mathcal{L}) \cap Q(V)$ is dense in $L^2(\rho)$;

(ii) the potential V is U^p-regular for some (all) $1 \le p < \infty$ iff $Q(\mathcal{L}) \cap Q(V)$ is a form core for $\mathcal{E}$.

In particular, if $V \in L^1_{\text{loc}}(\rho)$, then V is U^p-regular.

Proof. (i). First we assume that $Q(\mathcal{L}) \cap Q(V)$ is dense in $L^2(\rho)$. Let $\mathcal{S} := \mathcal{L} \dotplus V$. By Theorem 2.1.8 the sequence $\mathcal{L} + V_n \to \mathcal{S}$ in strong resolvent sense. Making use of Theorem 2.2.9 we conclude that $\exp(-t\mathcal{S}) = \text{s-}\lim\limits_n U_{V_n}(t)$, $t \ge 0$. Therefore V is U-admissible. Since the semigroups U^p, $1 \le p < \infty$, are consistent the statement follows from ([77], Prop. 3.1).

Now we suppose that the potential V is U^p-admissible. Therefore it is U-admissible, i.e. $\exp(-t(\mathcal{L} + V_n)) \to U_V(t)$, $t \ge 0$, strongly in $L^2(\rho)$. By Theorem 2.2.9 the sequence $\mathcal{L} + V_n \to \mathcal{L}_V$ in strong resolvent sense, where $-\mathcal{L}_V$ stands for the generator of U_V. Due to the operator inequality $\mathcal{L} + V_n \le \mathcal{L} + V_{n+1}$ we see that $\mathcal{L} + V_n \le \mathcal{L}_V$. Hence $Q(\mathcal{L} + V_n) \supset Q(\mathcal{L}_V)$ and

$$\langle \mathcal{L}\varphi, \varphi \rangle + \langle V_n\varphi, \varphi \rangle = \langle (\mathcal{L} + V_n)\varphi, \varphi \rangle \le \langle \mathcal{L}_V\varphi, \varphi \rangle$$

for all $\varphi \in Q(\mathcal{L}_V)$ and $n \in \mathbb{N}$. The monotone convergence theorem yields $Q(\mathcal{L}_V) \subset Q(\mathcal{L}) \cap Q(V)$.

(ii). The proof of (i) implies that $\mathcal{L}_{V-V_n} = \mathcal{L} \dotplus (V - V_n)$ for all $n \in \mathbb{N}$. Let s stand for the closure of $\mathcal{E} \restriction_{Q(\mathcal{L}) \cap Q(V)}$ and $\mathcal{S}$ for the associated self-adjoint operator. We note that for all $\varphi \in Q(\mathcal{L}_V)$ and $n \in \mathbb{N}$ the following estimate holds

$$\langle \mathcal{L}\varphi, \varphi \rangle + \langle (V - V_n)\varphi, \varphi \rangle \ge \langle \mathcal{L}\varphi, \varphi \rangle + \langle (V - V_{n+1})\varphi, \varphi \rangle \ge \langle \mathcal{L}\varphi, \varphi \rangle.$$

If the set $Q(\mathcal{L}) \cap Q(V)$ is a form core for $\mathcal{L}$ (i.e. $\mathcal{S} = \mathcal{L}$), then the convergence theorem for forms implies that $\mathcal{L}_{V-V_n} \to \mathcal{L}$ in strong resolvent sense and, hence, the potential V is U-regular. If we assume that V is U-regular then we conclude that $\mathcal{L} = S$ using the convergence theorem for forms. $\square$

The next statement shows that if the form method is applicable, then the semigroups, constructed by means of the form τ_Λ and the approach described in this subsection, coincide.

Lemma 2.3.7. *Let $p \geq 1$. Let V, $W \in L^1_{\text{loc}}(\rho)$, $V \geq 0$ and W be real-valued. We assume that $W \in PK_a(\mathcal{L} \dotplus V)$ with some $a > 0$. Let $(\mathcal{L}+V)_p$ and $\mathcal{N}_p^{\pm}$ stand for the generators of C_0-semigroups on $L^p(\rho)$, associated with the closures $\mathcal{E}$ and τ of the forms $\langle \nabla u,\, \nabla v\rangle + \langle Vu,\, v\rangle$ and $\langle \nabla u,\, \nabla v\rangle + \langle (V \pm iW)u,\, v\rangle$, $u, v \in C_0^1(\Omega)$, respectively. Then*

$$\exp(-t\mathcal{N}_p^{\pm}) = s\text{-}\lim_n \lim_m \exp\big(-t(\mathcal{L}_p + V_n + iW_m)\big).$$

Proof. For $n \in \mathbb{N}$ let $\mathcal{E}_n$ stand for the closure of the form

$$\langle \nabla u,\, \nabla v\rangle + \langle V_n u,\, v\rangle, \quad u, v \in C_0^1(\Omega).$$

It follows from Theorem 2.1.8 that $\mathcal{L}+V_n \to \mathcal{L} \dotplus V$ in $L^2(\rho)$ in the strong resolvent sense. By Theorem 2.2.9 we conclude that $\exp(-t(\mathcal{L}+V_n)) \to \exp(-t(\mathcal{L} \dotplus V))$ strongly in $L^2(\rho)$ for all $t \geq 0$. On the other hand, Proposition 2.3.2 implies that $\exp(-t(\mathcal{L}_p + V_n)) \to \exp(-t\mathcal{L}_p)_V$ strongly in $L^p(\rho)$. Since the family

$$U^p := \Big(\exp(-t(\mathcal{L} \dotplus V)) \restriction_{L^2(\rho)\cap L^p(\rho)}\Big)^{\sim}_{p\to p}, \quad 1 \leq p < \infty,$$

is consistent we conclude that

$$\exp(-t(\mathcal{L}+V)_p) = s\text{-}L^p(\rho)\text{-}\lim_n \exp(-t(\mathcal{L}_p + V_n)).$$

Let $n \in \mathbb{N}$ be fixed. For $m \in \mathbb{N}$ let $\tau_{n,m}$ stand for the closure of the form

$$\langle \nabla u,\, \nabla v\rangle + \langle (V_n + iW_m)u,\, v\rangle, \quad u, v \in C_0^1(\Omega).$$

It follows from Lemma 2.1.9 that $\mathcal{L} + V_n + W_m \to \mathcal{L} \dotplus V_n + iW$ in $L^2(\rho)$ in the strong resolvent sense. By Theorem 2.2.9 we conclude that $\exp(-t(\mathcal{L} + V_n + iW_m)) \to \exp(-t(\mathcal{L} \dotplus V_n + iW))$ strongly in $L^2(\rho)$ for all $t \geq 0$. On the other hand, Proposition 2.3.2 implies that $\exp(-t(\mathcal{L}_p + V_n + iW_m)) \to \exp(-t(\mathcal{L}_p + V_n)_W$ strongly in $L^p(\rho)$. Since the semigroups

$$T^p(t) := \Big(\exp(-t(\mathcal{L} \dotplus V + iW)) \restriction_{L^2(\rho)\cap L^p(\rho)}\Big)^{\sim}_{p\to p}, \quad 1 \leq p < \infty,\ t \geq 0,$$

are consistent the statement of the lemma follows. $\square$

Remark. *The assertion of Lemma 2.3.7 is still true if we assume that $V = V^+ - V^-$, with $V^{\pm} \geq 0$, and $V^- \in PK_\alpha(\mathcal{L} \dotplus V^+)$ for some $\alpha \in [0,1)$.*

2.3.3 Case of Non-trivial Drifts

Recently Z. Sobol and H. Vogt (see [72]) have developed another method of associating a C_0-semigroup on $L^p \equiv L^p(\Omega, dx)$ with the differential expression Λ, without assuming that the corresponding form is sectorial. We assume that μ is the Lebesgue measure and the potential $q = V$ is real-valued. Then Λ takes form

$$\Lambda := -\nabla \cdot a \cdot \nabla + b \cdot \nabla + V \tag{2.11}$$

Below we outline the idea of this approach.

Let the matrix $a \in L^1_{\text{loc}}$ (i.e. the functions $a_{kj} \in L^1_{\text{loc}}$ for all $k, j = 1, \dots, d$) be a.e. invertible with $a^{-1} \in L^1_{\text{loc}}$. We assume that the corresponding bilinear form is sectorial. We set $a_s := \dfrac{1}{2}(a + a^T)$, where a^T stands for the transpose of a. Then

$$\tau_N(u,\, v) = \langle \nabla u \cdot a \cdot \nabla v \rangle,\ \mathcal{D}(\tau_N) = \{u \in W^{1,1}_{\text{loc}} \cap L^2 : \nabla \overline{u} \cdot a_s \cdot \nabla u \in L^1\}$$

defines a closed sectorial Dirichlet form in L^2. Let $\tau_a \subset \tau_N$ be a Dirichlet form. Set $W := b \cdot a_s^{-1} \cdot b$ and assume that W and $|V|$ are τ_a-regular, i.e. $Q(W) \cap \mathcal{D}(\tau_a)$ and $Q(|V|) \cap \mathcal{D}(\tau_a)$ are cores for the form τ_a (see Proposition 2.3.6).

The form τ on $\mathcal{D}(\tau) := \mathcal{D}(\tau_a) \cap Q(|V| + W)$ is defined by

$$\tau(u,\, v) = \langle \nabla u \cdot a \cdot \nabla v \rangle + \langle b \cdot \nabla u,\, v \rangle + \langle V u,\, v \rangle.$$

This form is well-defined since the Schwarz inequality implies that

$$|\nabla u \cdot b\overline{v}| = \left|(a_s^{\frac{1}{2}} \nabla u) \cdot (a_s^{-\frac{1}{2}} b\overline{v})\right| \le (\nabla u \cdot a_s \cdot \nabla u)^{\frac{1}{2}} (W|v|^2)^{\frac{1}{2}} \in L^1 \tag{2.12}$$

for $u,\, v \in \mathcal{D}(\tau)$. It is also possible to show that $\mathcal{D}(\tau)$ is dense in $\mathcal{D}(\tau_a)$. Now introducing the large potential $U_0 := |W| + 2V^-$ one can see that the form $\tau_m := \tau + U - U \wedge m$ is closed and sectorial for all $U \ge U_0$ and $m \in \mathbb{N}$. Indeed, the sum of the first order terms is form small w.r.t. $\tau + W$ by (2.12). It is a remarkable fact that the form τ need not be sectorial. Since τ_m is sectorial one can follow a traditional procedure (the form method described in subsection 2.3.1) and construct the corresponding quasi-contractive semigroup. The next natural step is to pass to the limit as $m \to \infty$ and find the conditions under which this limit (which exists by the monotone convergence theorem) is a C_0-semigroup). We introduce the forms

$$\tau_p[u] = \frac{4}{pp'} \langle \nabla u \cdot a \cdot \nabla u \rangle + \frac{2}{p} \langle b \cdot \nabla u,\, u \rangle + \langle V |u|^2 \rangle,\ 1 < p < \infty,$$
$$\tau_1[u] = 2\langle b \cdot \nabla u,\, u \rangle + \langle V |u|^2 \rangle,$$

and formulate the relevant result.

Theorem 2.3.8. *Let $T_0 := T_0^2$ be the C_0-semigroup associated with the form $\tau + U_0$ on L^2. Let I be the set of all $p \in [1, \infty)$ such that $\tau_p \geq -k_p$ for some $k_p \in \mathbb{R}$.*

(i) Then I is an interval in $[1, \infty)$, which contains 2, and T_0 extrapolates to a C_0-semigroup T_0^p on L^p with the generator $\mathcal{A}_{0,p}$ for all $p \in I$.

(ii) The sequence of C_0-semigroups T_m^p generated by $\mathcal{A}_{0,p} - U_0 \wedge m$, $m \in \mathbb{N}$, strongly converges to a quasi-contractive C_0-semigroup T^p. For $p \in I$, the semigroups T^p are consistent and $\|T^p\|_{L^p \to L^p} \leq \exp(\kappa_p t)$.

(iii) If, in addition, we assume that

$$\left|\operatorname{Im}\langle bu, \nabla u\rangle\right| \leq c_1 \tau_p[u] + c_2 \|u\|_2^2 \quad \textit{for all} \quad u \in \mathcal{D}(\tau),$$

for some $p \in \operatorname{Int} I$, $c_1 \geq 0$, $c_2 \in \mathbb{R}$, then the semigroup T^p extends to an analytic semigroup on L^p for all $p \in \operatorname{Int} I$, where $\operatorname{Int} I$ stands for the interior of I.

2.4 Problem of Strong Uniqueness

A substantial part of this research is devoted to studying the strong uniqueness problem for elliptic operators. In the present section we introduce the notion of uniqueness for generators of C_0-semigroups and discuss different methods of treating this problem.

2.4.1 Notion of Uniqueness

Let $T(t)$, $t \geq 0$, be a C_0-semigroup on a Banach space X. By $\mathcal{A}$ we denote its generator. We start with the following definition.

Definition 2.4.1. *Let $\mathcal{D}_0 \subset \mathcal{D}(\mathcal{A})$ and $\mathcal{A}_0 := \mathcal{A} \restriction_{\mathcal{D}_0}$. The set $\mathcal{D}_0$ is said to be a domain of strong uniqueness for the operator $\mathcal{A}$ if $\mathcal{A}$ is the only closed extension of $\mathcal{A}_0$ which generates a C_0-semigroup on X.*

Theorem 2.4.3 below establishes a relation between domains of strong uniqueness for generators and their cores (see e.g. [61], Th. A-II.1.31). First we formulate an auxiliary result.

Theorem 2.4.2. *Let $\mathcal{A}$ be the generator of a C_0-semigroup on X. We assume that the linear operator $\mathcal{B}$ is continuous on $\mathcal{D}(\mathcal{A})$ in the graph norm of $\mathcal{A}$. Then the operator $\mathcal{A}+\mathcal{B}$ on $\mathcal{D}(\mathcal{A}+\mathcal{B})=\mathcal{D}(\mathcal{A})$ generates a C_0-semigroup on X.*

Theorem 2.4.3. *The set $\mathcal{D}_0$ is a domain of strong uniqueness for the operator $\mathcal{A}$ if and only if it is a core of $\mathcal{A}$, i.e. $\mathcal{A}=\overline{\mathcal{A}\restriction_{\mathcal{D}_0}}$.*

Proof. First assume that $\mathcal{D}_0$ is a core of $\mathcal{A}$. Let $\mathcal{B}$ be the generator of a C_0-semigroup such that $\mathcal{B}\supset\mathcal{A}_0$. Then $\mathcal{B}\supset\mathcal{A}$ since $\mathcal{A}$ is the closure of $\mathcal{A}_0$. Since $\rho(\mathcal{A})\cap\rho(\mathcal{B})\neq\emptyset$ we readily see that $\mathcal{A}=\mathcal{B}$.

Now we assume that $\mathcal{D}_0$ is not a core of the operator $\mathcal{A}$, i.e. $\mathcal{D}_0$ is not dense in $\mathcal{D}(\mathcal{A})$ w.r.t. the graph norm of $\mathcal{A}$. Then there exists a non-zero linear continuous functional ϕ in $\mathcal{D}(\mathcal{A})$ such that $\phi\restriction_{\mathcal{D}_0}=0$. Let $u\in\mathcal{D}(\mathcal{A})$ fixed. We define a linear operator $\mathcal{B}$ on $\mathcal{D}(\mathcal{A})$ by $\mathcal{B}x=\phi(x)u$. The inequality

$$||\mathcal{B}x||_{\mathcal{D}(\mathcal{A})}\leq c||u||_{\mathcal{D}(\mathcal{A})}||x||\leq c_u||x||_{\mathcal{D}(\mathcal{A})},$$

where $||x||_{\mathcal{D}(\mathcal{A})}$ stands for the graph norm of $x\in\mathcal{D}(\mathcal{A})$, implies that the operator $\mathcal{B}$ is bounded in $\mathcal{D}(\mathcal{A})$. Therefore by Theorem 2.4.2 the operator $\mathcal{A}+\mathcal{B}$ with the domain $\mathcal{D}(\mathcal{A})$ is the generator of a C_0-semigroup. Clearly, $\mathcal{A}+\mathcal{B}\restriction_{\mathcal{D}_0}=\mathcal{A}_0$ and $\mathcal{A}+\mathcal{B}\neq\mathcal{A}$ if $u\neq 0$. Hence, $\mathcal{D}_0$ is not a domain of strong uniqueness. □

Let $(M,\mathcal{M},\mu)$ be a measure space. Let $\mathcal{A}$ be a symmetric operator in $L^2(M,\mu)$ with the domain $\mathcal{D}(\mathcal{A})$. Definition 2.4.1 and Theorem 2.4.3 imply that the $\mathcal{D}_0$ is a domain of strong uniqueness for the operator $\mathcal{A}$ iff $\mathcal{A}$ is essentially self-adjoint on $\mathcal{D}_0$.

Let $q:\mathbb{R}^d\to\mathbb{C}$ and $q\in L^2_{\mathrm{loc}}(\mathbb{R}^d)$. In what follows we discuss different techniques to prove essential self-adjointness (m-accretive closability) for the operator $-\Delta+q\restriction_{C_0^\infty(\mathbb{R}^d)}$.

2.4.2 Kato-Rellich Theorem

In order to describe the first method we need the following definition.

Definition 2.4.4. *(see e.g. [64, Th. X.2]). Let $\mathcal{A}$ and $\mathcal{B}$ be densely defined operators in the Hilbert space X. We assume that $\mathcal{D}(\mathcal{B})\supset\mathcal{D}(\mathcal{A})$. Let $a,b\in\mathbb{R}$ be constants such that for all $\varphi\in\mathcal{D}(\mathcal{A})$*

$$||\mathcal{B}\varphi||\leq a||\mathcal{A}\varphi||+b||\varphi||. \tag{2.13}$$

Then the operator $\mathcal{B}$ is said to be $\mathcal{A}$-bounded. The infimum of all a which satisfy (2.13) *is called the $\mathcal{A}$-bound of the operator $\mathcal{B}$. If the $\mathcal{A}$-bound equals* 0 *the operator $\mathcal{B}$ is said to be infinitesimally small w.r.t. $\mathcal{A}$.*

The following result known as the Kato-Rellich theorem is of fundamental importance for many problems in the perturbation theory.

Theorem 2.4.5. *(see e.g. [64, Th. X.12]). Suppose that $\mathcal{A}$ is self-adjoint, $\mathcal{B}$ is symmetric, and $\mathcal{B}$ is $\mathcal{A}$-bounded with relative bound $a < 1$. Then the operator $\mathcal{A}+\mathcal{B}$ is self-adjoint on $\mathcal{D}(\mathcal{A})$ and essentially self-adjoint on any core of $\mathcal{A}$. Furthermore, if $\mathcal{A} \geq m\,\mathrm{Id}$ then $\mathcal{A}+\mathcal{B}$ is bounded below by $\left(m - \frac{b}{1-a}\right) \vee \left(a|m| + b\right)$, where a, b are given by* (2.13).

Next we see how Theorem 2.4.5 can be employed to establish essential self-adjointness of $-\Delta + V \restriction_{C_0^\infty}$, where V is the operator of multiplication by a measurable real-valued function V.

Let $(M, \mathcal{M}, \mu)$ be a measure space. Let $L^r(\mu) + L^s(\mu)$ stand for the set of measurable functions $f = f_1 + f_2$, where $f_1 \in L^r(\mu)$ and $f_2 \in L^s(\mu)$.

Theorem 2.4.6. *Let $M = \mathbb{R}^3$ and μ is the Lebesgue measure. We set $L^p := L^p(dx)$, $p \geq 1$. Let $V \in L^2 + L^\infty$ be real-valued. Then the operator $-\Delta + V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$ and self-adjoint on $\mathcal{D}(-\Delta)$.*

Proof. Since the potential V is real-valued the corresponding operator of multiplication is self-adjoint on

$$\mathcal{D}(V) := \{\varphi \in L^2 \,|\, V\varphi \in L^2\}.$$

Making use of the representation $V = V_1 + V_2$ and the Hölder inequality we obtain the following estimate:

$$\|V\varphi\|_2 \leq \|V_1\|_2\|\varphi\|_\infty + \|V_2\|_\infty\|\varphi\|_2, \; \varphi \in C_0^\infty(\mathbb{R}^3) \subset \mathcal{D}(V). \tag{2.14}$$

By [64, IX.28], for every $\varepsilon > 0$ there is a constant $b(\varepsilon) \geq 0$ such that

$$\|\varphi\|_\infty \leq \varepsilon\|\Delta\varphi\|_2 + b(\varepsilon)\|\varphi\|_2$$

for all $\varphi \in C_0^\infty(\mathbb{R}^3)$. Hence estimate (2.14) implies that

$$\|V\varphi\|_2 \leq \varepsilon\|V_1\|_2\|\varphi\|_2 + \left(b(\varepsilon) + \|V_2\|_\infty\right)\|\varphi\|_2$$

for all $\varphi \in C_0^\infty(\mathbb{R}^3)$. Taking $\varepsilon < \|V_1\|_2^{-1}$ and applying the Kato-Rellich theorem we complete the proof. □

Next we present a self-adjointness result in higher dimensions.

Theorem 2.4.7. *([64, Th. X.20]). Let $d \geq 4$ and $V \in L^p(\mathbb{R}^d)$ with $p > d/2$. Then V is infinitesimally small w.r.t. the operator $-\Delta$.*

Proof. It follows from ([64, Th. IX.27]) that if $u \in \mathcal{D}(-\Delta)$, then the function

$$(1+\lambda^2)\hat{u}(\lambda) \in L^2(\mathbb{R}^d),$$

where $\hat{u}$ stands for the Fourier transform of u. The assumption $p > d/2$ yields $(1+\lambda^2)^{-1} \in L^p(\mathbb{R}^d)$ and the Hölder inequality implies that $\hat{u} \in L^q(\mathbb{R}^d)$ and

$$\|\hat{u}\|_q \leq \|(1+\lambda^2)^{-1}\|_p \|(1+\lambda^2)\hat{u}\|_2,$$

where $q^{-1} = p^{-1} + \frac{1}{2}$. Therefore the Hausdorff-Young inequality implies that $u \in L^r(\mathbb{R}^d)$, where $f^{-1} = \frac{1}{2} - p^{-1}$. Making use of the assumption $V \in L^p(\mathbb{R}^d)$ and the Hölder inequality we see that $Vu \in L^2(\mathbb{R}^d)$. Hence, $\mathcal{D}(V) \supset \mathcal{D}(-\Delta)$ and

$$\begin{aligned}
\|Vu\|_2 &\leq \|V\|_p \|u\|_r \leq \|V\|_p \|\hat{u}\|_q \\
&= \|V\|_p \|(1+\nu\lambda^2)^{-1}(1+\nu\lambda^2)\hat{u}\|_q \\
&\leq \|V\|_p \|(1+\nu\lambda^2)^{-1}\|_p \|(1+\nu\lambda^2)\hat{u}\|_2 \\
&\leq \left(\|V\|_p \|(1+\lambda^2)^{-1}\|_p\right) \nu^{-\frac{s}{2p}} \left(\|u\|_2 + t\|\Delta u\|_2\right).
\end{aligned}$$

Taking into account that $p > d/2$ we complete the proof. □

2.4.3 Kato Inequality. Essential Self-adjointness and J-Self-adjointness of Schrödinger-type Operators

Another powerful method to establish the strong uniqueness for the Schrödinger operators is based on a certain inequality for distributions. In order to formulate the relevant results we need the following definition.

Definition 2.4.8. *Let f be a distribution. We say that f is non-negative if $\langle f, \varphi \rangle$ for every $\varphi \in C_0^\infty$, $\varphi \geq 0$. If f and g are distributions and $f - g \geq 0$, we write $f \geq g$ in the sense of distributions.*

Next we state the *Kato inequality.*

Theorem 2.4.9. *(see e.g. [64, Th. X.27]). Let $u \in L^1_{\text{loc}}(\mathbb{R}^d)$ and its distributional Laplacian $\Delta u \in L^1_{\text{loc}}(\mathbb{R}^d)$. Then the following inequality holds in the distributional sense:*

$$\Delta|u| \geq \operatorname{Re}\left[(\operatorname{sgn}\bar{u})\Delta u\right],$$

(recall that $\operatorname{sgn} u(x) = u(x)|u(x)|^{-1}$ *if* $u(x) \neq 0$, *and* $\operatorname{sgn} u(x) = 0$ *otherwise).*

Below we see how Theorem 2.4.9 can be employed in order to prove uniqueness results for the Schrödinger operators. The following theorem is a simple extension of Theorem X.28 in [64].

Theorem 2.4.10. *Let V, $W \in L^2_{\rm loc}(\mathbb{R}^d)$ and $V \geq 0$. Then the closure in $L^2(\mathbb{R}^d)$ of the operator $\mathcal{H} = -\Delta + V + iW \restriction_{C_0^\infty}$ is m-accretive.*

Proof. One can readily see that the operator $\mathcal{H}$ is accretive. Therefore, by the Lumer-Philips theorem (Theorem 2.2.8), it suffices to show that the range of $1+\mathcal{H}$ is dense in $L^2(\mathbb{R}^d)$, i.e. that if $u \in L^2(\mathbb{R}^d)$ and

$$\langle (1+\mathcal{H})\varphi,\, u\rangle = 0 \quad \text{for all} \quad \varphi \in C_0^\infty(\mathbb{R}^d),$$

then $u = 0$. The latter implies that

$$(1+\mathcal{H})^* u = 0,$$

since $C_0^\infty(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$. Equivalently, we have

$$(1 - \Delta + V - iW)u = 0, \quad u \in L^2(\mathbb{R}^d),$$

where the derivative Δu is understood in the sense of distributions. Hence, $\Delta u = u + Vu \in L^1_{\rm loc}(\mathbb{R}^d)$. So by Theorem 2.4.9 we obtain

$$\Delta|u| \geq \operatorname{Re}(\operatorname{sgn} u\,)\Delta u) = \operatorname{Re}(1 + V - iW)|u| = (1+V)|u| \geq 0. \tag{2.15}$$

For $\delta > 0$ by J_δ we denote the universal mollifier, i.e. $J_\delta(x) = J(x/\delta)\delta^{-d}$, where $J \in C_0^\infty$, $J \geq 0$ and $\langle J\rangle = 1$. Set $w := |u|$, $w_\delta := w * J_\delta$. Then $\Delta w_\delta = w * \Delta J_\delta \in L^2(\mathbb{R}^d)$, so $w_\delta \in \mathcal{D}(\Delta)$ and $\langle w_\delta,\, \Delta w_\delta\rangle \leq 0$, with equality only if $w_\delta = 0$. But, on the other hand, $\Delta w_\delta = \Delta * J_\delta \geq 0$ by (2.15) and so $\Delta w_\delta \geq 0$ pointwise. Therefore $\langle w_\delta,\, \Delta w_\delta\rangle \geq 0$ and $w_\delta = 0$. Since $w_\delta \to w$ in $L^2(\mathbb{R}^d)$ it follows that $w = |u| = 0$. □

Remark. *It is easy to see that Theorem 2.4.10 is still valid if we assume that V is bounded below.*

The proof of the next theorem is similar to that of Theorem X.29 in [64] and therefore omitted.

Theorem 2.4.11. *Let $V = V_1 + V_2$, where $V_1 \in L^2_{\rm loc}(\mathbb{R}^d)$, $V_1 \geq 0$ and V_2 is $-\Delta$-bounded with the relative bound less than 1. Let $W \in L^2_{\rm loc}(\mathbb{R}^d)$. Set $q := V + iW$. Then the closure in $L^2(\mathbb{R}^d)$ of $-\Delta + q \restriction_{C_0^\infty}$ is m-accretive.*

In the next statement the conditions on V_1 and V_2 are less restrictive. The result is due to T. Kato ([34], Theorem).

Let $Q^* : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function such that $Q^*(r) = o(r^2)$ as $r \to \infty$. Let $V \in L^2_{\text{loc}}(\mathbb{R}^d)$, $V = V_1 + V_2$, where V_1 and V_2 satisfy the following conditions.

(i) The potential $V_1 \in L^2_{\text{loc}}(\mathbb{R}^d)$ and $V_1(x) \geq Q^*(|x|)$, $x \in \mathbb{R}^d$;

(ii) There exist positive numbers K and s such that

$$\int\limits_{|x|\leq r} V_2^2(x)dx \leq Kr^{2s},$$

and

$$\lim_{\rho\to 0} \sup_{x\in\mathbb{R}^d} \int\limits_{|x-y|\leq\rho} |V_2(y)||x-y|^{2-d}dy = 0, \tag{2.16}$$

where $|x-y|^{2-d}$ is be replaced by $1 - \log|x-y|$ if $d = 2$, and by 1 if $d = 1$;

(ii') If $d \geq 5$ condition (ii) may be replaced by $V_2 \in L^{d/2}(\mathbb{R}^d)$.

Theorem 2.4.12. *The operator $-\Delta + V \restriction_{C_0^\infty}$ is essentially self-adjoint if conditions (i) and (ii) are fulfilled. In the case $d \geq 5$ (ii) may be replaced by (ii').*

Remark. *If a potential V satisfies* (2.16) *we say that it belongs to the Kato class. For more details on the Kato class see subsection 2.6.2.*

Next we discuss how the assumptions on the potential V can be weakened further, with the essential self-adjointness being preserved. Proofs of Theorems 2.4.13, 2.4.14, 2.4.16 below can be found in [18, 24, 25].

(iii) We assume that there exist numbers $(a_n, b_n)_{n\in\mathbb{N}} \subset (0, \infty)$ such that the annuli

$$S_n := \{x \in \mathbb{R}^d \mid a_n \leq |x| \leq b_n\},\ n \in \mathbb{N}$$

are disjoint, and there is a non-decreasing function $Q^* : \cup_{n\in\mathbb{N}}[a_n, b_n] \to \mathbb{R}_+$ satisfying the following conditions:

$$\sum_{n\geq 1} \int_{a_n}^{b_n} \frac{dr}{\sqrt{Q^*(r)}} < \infty \quad \text{and} \quad (b_n - a_n)^2 \geq \frac{K}{Q^*(b_n)}$$

for some $K > 0$.

The relevant result on the essential self-adjointness reads as follows.

Theorem 2.4.13. *Let $V \in L^2_{\mathrm{loc}}$. We assume that $V = V_1 + V_2$ where V_2 satisfies (ii) (or (ii') if $d \geq 5$), and $V_1(x) \geq Q^*(|x|)$ for all $x \in \cup_{n\in\mathbb{N}} S_n$, where Q^* and S_n, $n \in \mathbb{N}$, are as in condition (iii). Then the operator $-\Delta + V \restriction_{C_0^\infty}$ is essentially self-adjoint.*

In order to produce an example of a potential V such that the essential self-adjointness of $-\Delta + V \restriction_{C_0^\infty}$ is lost we need to introduce some new notions.

Let $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$. By $\mathcal{N}$ we denote an operator in $L^2(\mathbb{R}_+)$, defined by

$$\mathcal{N}u = \frac{d^2u}{dx^2} + Vu, \quad u \in \mathcal{D}(\mathcal{N}),$$

where

$$\mathcal{D}(\mathcal{N}) := \{u \in L^2(\mathbb{R}_+) \,|\, u' \in AC(\mathbb{R}_+),\, u(0)\cos(\alpha) + u'(0)\sin(\alpha) = 0\}.$$

We say that the operator $\mathcal{N}$ is *limit-circle*, if all solutions of the equation $\mathcal{N}u = 0$ are in $L^2(\mathbb{R}_+)$. If at least one solution is not in $L^2(\mathbb{R}_+)$, then the operator $\mathcal{N}$ is said to be *limit-point.* Some criteria for $\mathcal{N}$ to be limit-point or limit-circle as well as numerous references can be found in [18] (see also [21, Section III.10]).

The following theorem gives a sufficient condition ensuring that $-\Delta + V \restriction_{C_0^\infty}$ is not essentially self-adjoint.

Theorem 2.4.14. *Let $V \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$. Let Ω_{d-1} be some open bounded set in $\mathbb{R}^{d-1}$. We assume that $V = V(x_1)$ in the tube $\mathbb{R}_+ \times \Omega_{d-1}$ and the operator $\mathcal{N} = \dfrac{d^2u}{dx^2} + Vu$, $u \in \mathcal{D}(\mathcal{N})$, is limit-circle in $L^2(\mathbb{R}^+)$. Then $-\Delta + V \restriction_{C_0^\infty}$ is not essentially self-adjoint regardless of the definition of V elsewhere in $\mathbb{R}^d$.*

Next we discuss an extension of the notion of self-adjointness to non-symmetric operators and indicate its relation to that of m-accretivity.

An operator J on a Hilbert space $\mathcal{X}$ is called a *conjugation* operator if

$$\langle Jx,\, Jy\rangle = \langle y,\, x\rangle, \quad J^2x = x,$$

for all $x, y \in \mathcal{X}$, where $\langle\cdot,\, \cdot\rangle$ stand for the inner product in X. It follows from the definition that

$$\langle Jx,\, y\rangle = \langle Jy,\, x\rangle \text{ for all } x, y \in \mathcal{X}.$$

Furthermore, if $\mathcal{A}$ is a densely defined linear operator in $\mathcal{X}$. Then

$$(J\mathcal{A}J)^* = J\mathcal{A}^*J.$$

Let J be a conjugation operator on $\mathcal{X}$. A densely defined linear operator $\mathcal{A}$ is said to be *J-symmetric* (*-self-adjoint*), if

$$(J\mathcal{A}J)^* \subset (=)\mathcal{A}^*.$$

Essentially J-self-adjoint operators are defined analogously. It appears that for J-self-adjoint operators one can develop a theory similar to the self-adjoint case (for details see e.g. [21, Section III.5]). Here we only formulate a simple statement which reveals the relation between J-self-adjoint and m-accretive operators in L^2.

Theorem 2.4.15. *(see e.g. [21, Th. III.6.7]) A closed J-symmetric operator $\mathcal{A}$ is m-accretive iff $\mathcal{A}$ is J-self-adjoint and accretive.*

We complete this subsection by stating a result on J-self-adjointness for the Schrödinger operators. Theorem 2.4.16 below can be obtained as a consequence of [25, Theorems 6 and 7].

Theorem 2.4.16. *Let $V, W \in L^2_{\text{loc}}(\mathbb{R}^d)$. We assume that there are constants $a > 0$ and $c_1, c_2 \in \mathbb{R}$ such that $V \geq -c_1^2|x|^\alpha$ and $W(x) = \pm c_2^2|x|^\beta$, if $|x| \geq a$. Then the operator $-\Delta + V + iW \restriction_{C_0^\infty}$ is essentially J-self-adjoint, if any one of the following conditions holds:*

(i) $\beta \leq 0$ and $\alpha \leq 2$;

(ii) $\beta > 0$ and $\alpha < 2\beta + 2$;

(iii) $c_2^2 > c_1\beta$ and $\alpha = 2\beta + 2$.

2.4.4 Wienholtz Method

Next we discuss another approach to establishing the strong uniqueness for second order differential operators. It is this technique that is employed in Chapter 3 to investigate the perturbations of the Dirichlet operators by lower order terms. Sometimes this approach is called the Wienholz method. In order to illustrate the main idea of the approach we provide a proof of the essential self-adjointness in $L^2(\mathbb{R}^d)$ for the operator $-\Delta \restriction_{C_0^\infty}$.

Let $\mathcal{F}$ stand for the class of smooth, compactly supported, spherically symmetric functions η such that $0 \le \eta \le 1$ and $\eta = 1$ on some ball centered at the origin.

Let $\eta \in \mathcal{F}$. By $\mathcal{L}_\eta$ we denote the operator in $L^2(\mathbb{R}^d) =: L^2$ associated with the closure of the following form:

$$\mathcal{E}_\eta(u,v) := \langle \eta\nabla u,\, \eta\nabla v\rangle, \quad u,\, v \in C_b^1(\mathbb{R}^d).$$

One can easily see that $\mathcal{L}_\eta \restriction_{C_b^2\cap L^2} = -\nabla\cdot\eta^2\nabla$.

First we prove the following conditional result.

Theorem 2.4.17. *We assume that for every $\eta \in \mathcal{F}$ the operator $\mathcal{L}_\eta \restriction_{C_0^\infty}$ is essentially self-adjoint. Then C_0^∞ is an essential domain for the operator $-\Delta$.*

Proof. The operator $-\Delta \restriction_{C_0^\infty}$ is clearly non-negative. Therefore by the Lumer-Philips theorem (Theorem 2.2.8) in order to prove its essential self-adjointness it suffices to check that the range of $(1-\Delta)\restriction_{C_0^\infty}$ is dense in L^2, i.e. that the equality $\langle(1-\Delta)\varphi,\, u\rangle = 0$ with $u \in L^2$, which holds for all $\varphi \in C_0^\infty$, implies that $u = 0$. The rest of the proof is divided into three steps.

Step 1. Let $\eta,\, \xi \in \mathcal{F}$ and $\eta = 1$ on $\operatorname{supp}\xi$. Then

$$u\xi \in \mathcal{D}(\mathcal{L}_\eta^{\frac{1}{2}}) =: \mathcal{D}.$$

Let φ be an arbitrary element from C_0^∞. Since $\eta = 1$ on $\operatorname{supp}\xi$ we get

$$\langle(1-\Delta)\varphi,\, u\xi\rangle = \langle(1+\mathcal{L}_\eta)\varphi,\, u\xi\rangle. \tag{2.17}$$

The function $\xi\varphi \in C_0^\infty$, so our assumption yields

$$\langle(1-\Delta)(\xi\varphi),\, u\rangle = 0.$$

The last equality can be rewritten in the form

$$\langle(1-\Delta)\varphi,\, u\xi\rangle = \langle 2\nabla\xi\cdot\nabla\varphi + (\Delta\xi)\varphi,\, u\rangle. \tag{2.18}$$

Combining (2.17) and (2.18) we conclude that

$$\langle(1+\mathcal{L}_\eta)\varphi,\, u\xi\rangle = 2\langle\nabla\xi\cdot\nabla\varphi,\, u\rangle + \langle(\Delta\xi)\varphi,\, u\rangle. \tag{2.19}$$

We estimate the right-hand side of (2.19) using the Hölder inequality and taking into account that $\eta = 1$ on $\operatorname{supp}\xi$:

$$\|\nabla\xi \cdot \nabla\varphi\|_2 \le C_\xi \| \eta |\nabla\varphi| \|_2 = C_\xi \| \mathcal{L}_\eta^{\frac{1}{2}} \varphi\|_2,$$
$$\|(\Delta\xi)\varphi\|_2 \le C_\xi \|\varphi\|_2.$$

Hence, by the Schwarz inequality

$$|\langle (1+\mathcal{L}_\eta)\varphi,\, u\xi\rangle| \le C_{u,\xi} \|(1+\mathcal{L}_\eta)^{\frac{1}{2}}\varphi\|_2. \tag{2.20}$$

Estimate (2.20) implies that the left-hand side of (2.19) defines a linear continuous functional on $\mathcal{D}$, i.e. there is an element $v \in \mathcal{D}$ such that

$$\langle (1+\mathcal{L}_\eta)\varphi,\, u\xi\rangle = \langle (1+\mathcal{L}_\eta)^{\frac{1}{2}}\varphi,\, (1+\mathcal{L}_\eta)^{\frac{1}{2}} v\rangle.$$

The second representation theorem (Theorem 2.1.6) yields

$$\langle (1+\mathcal{L}_\eta)\varphi,\, u\xi\rangle = \langle (1+\mathcal{L}_\eta)\varphi,\, v\rangle,$$

and making use of the essential self-adjointness of $\mathcal{L}_\eta \restriction_{C_0^\infty}$ we obtain the equality

$$\langle \psi,\, u\xi\rangle = \langle \psi,\, v\rangle,$$

for all $\psi \in L^2$. Thus $v = u\xi$ a.e.

Step 2. Now we claim that for all $\xi \in \mathcal{F}$ the following inequality holds.

$$\|u\xi\|_2 \le \|u|\nabla\xi|\|_2.$$

Let us choose $\hat{\xi}, \eta \in \mathcal{F}$ such that $\hat{\xi} \restriction_{supp\xi} = 1$ and $\eta \restriction_{supp\hat{\xi}} = 1$. Set $\hat{u} := u\hat{\xi}$ and note that $\hat{u} \restriction_{supp\xi} = u \restriction_{supp\xi}$. By Step 1 $\hat{u} \in \mathcal{D}$.

Let $(\varphi_k)_{k\in\mathbb{N}} \subset C_0^\infty$ be a sequence such that $\varphi_k \to u\xi$ in $\mathcal{D}$. Due to the choice of ξ, $\hat{\xi}$ and η we have $u = \hat{u}$ and $\eta\nabla\hat{u} = \nabla\hat{u}$ on $\operatorname{supp}\xi$. We rewrite (2.19) with $\varphi = \varphi_k$:

$$\langle \nabla\varphi_k,\, \nabla(u\xi)\rangle = \langle \nabla\varphi_k,\, u\nabla\xi\rangle - \langle \varphi_k,\, \nabla\hat{u}\nabla\xi\rangle. \tag{2.21}$$

We pass to the limit as $k \to \infty$ in (2.21) and obtain

$$\|u\xi\|_2^2 + \|\nabla(u\xi)\|_2^2 = -\langle u\xi\nabla\xi,\, \nabla\hat{u}\rangle + \langle \nabla\xi \cdot \nabla(u\xi)\rangle.$$

Observing that $\nabla(u\xi) = u\nabla\xi + \xi\nabla\hat{u}$ and $\|\nabla(u\xi)\|_2 \ge 0$ we complete the proof of the claim.

Step 3. $u = 0$.

We choose a sequence (ξ_n) such that $\xi_n \to 1$ pointwise and $|\nabla \xi_n| \le 1$ to see that $u\xi_n \to u$ and $u\nabla\xi_n \to 0$ in L^2 which implies that $u = 0$. $\square$

In order to establish the essential self-adjointness of $-\Delta \upharpoonright_{C_0^\infty}$ it remains to show that C_0^∞ is a core for $\mathcal{L}_\eta$. Before formulating the relevant result we recall some properties of degenerate elliptic operators with smooth coefficients. For $\eta \in \mathcal{F}$ let Ω stand for the interior of $\operatorname{supp} \eta$. It is known (see e.g. [73, Th. 1]) that the closure $\bar{\mathcal{A}}_\eta$ of the operator $\mathcal{A}_\eta = -\nabla \cdot \eta^2 \cdot \nabla \upharpoonright_{C^2(\overline{\Omega})}$ generates a Feller semigroup (i.e. a C_0-semigroup of contractions on $C(\overline{\Omega})$). Hence, by the Lumer-Philips theorem, for all $\lambda > 0$ the set $\operatorname{Ran}(\lambda + \mathcal{A}_\eta) =: R_\eta$ is dense in $C(\overline{\Omega})$.

Theorem 2.4.18. *Let $\eta \in \mathcal{F}$. Then the operator $\mathcal{L}_\eta \upharpoonright_{C_0^\infty}$ is essentially self-adjoint.*

Proof. Let $\widehat{\mathcal{L}}_\eta$ be the non-negative self-adjoint operator associated with the closure of the form $\langle \eta\nabla u, \eta\nabla v\rangle$, $u, v \in C_0^1(\Omega)$, in $L^2(\Omega)$. (Note that here $\langle f, g\rangle = \int_\Omega f(x)g(x)\,dx$).

First we prove that $C^2(\overline{\Omega})$ is an essential domain for the operator $\widehat{\mathcal{L}}_\eta$.

We need to show that any element from the core of $\widehat{\mathcal{L}}_\eta$ can be approximated by functions from $C^2(\overline{\Omega})$ in the graph norm of $\widehat{\mathcal{L}}_\eta$. The operator $-\widehat{\mathcal{L}}_\eta$ is the generator of a C_0-semigroup on $L^2(\Omega)$. Therefore the set $(\lambda + \widehat{\mathcal{L}}_\eta)^{-1}C(\overline{\Omega})$ is a core of $\widehat{\mathcal{L}}_\eta$ for all $\lambda > 0$. Let $f \in C(\overline{\Omega})$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions from R_η such that $f_n \to f$ in $C(\overline{\Omega})$ (and in $L^2(\Omega)$ since the measure of Ω is finite). Therefore the operator $\widehat{\mathcal{L}}_\eta \upharpoonright C^2(\overline{\Omega})$ is essentially self-adjoint.

Hence, $\operatorname{Ran}(1 + \widehat{\mathcal{L}}_\eta) \upharpoonright_{C^2(\overline{\Omega})}$ is dense in $L^2(\Omega)$. This implies that $\operatorname{Ran}(1 + \mathcal{L}_\eta) \upharpoonright_{C_b^2 \cap L^2}$ is dense in L^2. Indeed, let $\varepsilon > 0$. For $f \in L^2$ we choose $v_1 \in C^2(\overline{\Omega})$ in such way that $\|[(1+\widehat{\mathcal{L}}_\eta)v_1 - f]\mathbb{1}_\Omega\|_2 < \varepsilon/2$. Let $v \in C_b^2 \cap L^2$ and v is an extension of v_1. Let $w \in C_0^2(\Omega^c)$ and $\|w - (v-f)\mathbb{1}_\Omega^c\|_2 < \varepsilon/2$. Then $v - w \in C_b^2 \cap L^2$ and

$$\|(1+\widehat{\mathcal{L}}_\eta)(v+w) - f\| \le \|[(1+\widehat{\mathcal{L}}_\eta)v_1 - f]\mathbb{1}_\Omega\|_2 + \|w - (v-f)\mathbb{1}_\Omega^c\|_2 < \varepsilon.$$

In order to complete the proof of the theorem we employ the standard approximation for functions from C_b^2 by elements of C_0^∞. $\square$

2.5 Linear Topological Spaces

In this section we collect some topological concepts which appear in Chapter 4.

Let (X, τ) be a topological space. We say that the topological space (X, τ) is *Hausdorff* if for every $x, y \in X$, $x \neq y$, one can find open sets U_x, $U_y \in \tau$ such that

$$x \in U_x, \quad y \in U_y, \quad \text{and} \quad U_x \cap U_y = \emptyset.$$

A family $\sigma \subset \tau$ is called a *base* for topology τ if for every $U \in \tau$ one can find sets $(U_\alpha)_{\alpha \in I} \subset \sigma$ such that

$$U = \cup_{\alpha \in I} U_\alpha.$$

A topological space (X, τ) is said to be a *linear topological space* if X is a linear space and the operations of summation and multiplication by a scalar are continuous.

A linear topological space is said to be *locally convex* if it has a base for its topology, consisting of convex sets.

Let (X, τ) be a topological space. Let $A, B \subset X$. A linear functional $f : X \to \mathbb{R}$ is said to separate the sets A and B if there exists a constant $c \in \mathbb{R}$ such that

$$f(x) \geq c \ \text{ for all } \ x \in A, \quad \text{and} \quad f(y) \leq c \ \text{ for all } \ y \in B.$$

A family $(f_\alpha)_{\alpha \in I}$ of linear functionals on X is said *to separate points* if for every $x \in X$ and every neighbourhood U_x of x one can find an $\alpha_x \in I$ such that

$$f_{\alpha_x}(x) > 0 \quad \text{and} \quad f_{\alpha_x} \restriction_{X \setminus U_x} = 0.$$

A topological space (X, τ) is called *Souslin* if it is linearly ordered, not separable but satisfies the *countable chain* condition, i.e. every family $(U_\alpha)_{\alpha \in I} \subset \tau$ such that

$$U_\alpha \cap U_\beta = \emptyset \ \text{ for all } \ \alpha \neq \beta,$$

is countable.

Let (X_α, τ_α), $\alpha \in I$, be a family of topological spaces. Let $X := \times_{\alpha \in I} X_\alpha$. We say that a topology τ, defined by

$$\tau = \{U = \times_{k=1}^{m} U_{\alpha_k} \times_{I \setminus \{\alpha_1, \ldots, \alpha_m\}} X_\alpha | \, U_{\alpha_j} \in \tau_{\alpha_j}\},$$

is called the *product topology* on X.

2.6 Fundamental Solutions of Parabolic and Elliptic Equations. Semi-linear Elliptic Inequalities in Exterior Domains

In this section we discuss some notions of the qualitative theory of linear parabolic and elliptic differential equations and semi-linear elliptic differential inequalities, related to the differential expression

$$\Lambda := \nabla \cdot a \cdot \nabla - b \cdot \nabla - V,$$

with measurable $b = b(x) = (b_j(x))_{j=1}^d$ and $V = V(x)$ and a uniformly elliptic symmetric matrix $a = a(x) = (a_{ij}(x))_{i,j=1}^d$, $x \in \mathbb{R}^d$, $d \geq 3$, with measurable entries. Recall that we use the notation

$$\nabla \cdot a \cdot \nabla = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial}{\partial x_j}\Big), \quad b \cdot \nabla = \sum_{j=1}^{d} b_j \frac{\partial}{\partial x_j}.$$

2.6.1 Parabolic Equations

Let $T > 0$. We consider the second order parabolic equation

$$\partial_t u(t,x) = \Lambda u(t,x) \tag{2.22}$$

in the domain $[0,T] \times \mathbb{R}^d$ (here and below $\partial_t = \frac{\partial}{\partial t}$). For a Banach space X let $L^p(Q;X)$, $1 \leq p \leq \infty$, $(C(Q;X))$ stand for the space of L^p-integrable (continuous) functions defined on a set Q and taking values in X.

We say that a function $u \in C([0,T];L^2(\mathbb{R}^d)) \cap L^2((0,T);H^1(\mathbb{R}^d))$ is a weak solution to the Cauchy problem

$$\partial_t u = \nabla \cdot a \cdot \nabla u - b \cdot \nabla u - Vu, \ 0 < t \leq T, \ u(0) = f \in L_c^\infty(\mathbb{R}^d),$$

if

$$\begin{gathered} b \cdot \nabla u \in L^1((0,T) \times \mathbb{R}^d), \ Vu \in L^1((0,T) \times \mathbb{R}^d), \\ \int_0^T \int_{\mathbb{R}^d} (\nabla u \cdot a \cdot \nabla \phi + \phi b \cdot \nabla u + V\phi u - u\partial_t \phi)\,dxdt = 0, \\ \forall\, \phi \in H_0^1((0,T);H^1(\mathbb{R}^d)) \cap L^\infty((0,T);L^\infty(\mathbb{R}^d)), \\ \lim_{t\to 0} \int_{\mathbb{R}^d} u(t,x)\psi(x)dx = \int_{\mathbb{R}^d} f(x)\psi(x)dx, \quad \forall \psi \in L^1 \cap L^\infty(\mathbb{R}^d). \end{gathered} \tag{2.23}$$

where H^1, H_0^1 and L_c^∞ stand for the Sobolev space of square integrable functions with the distributional derivatives in L^2, the space of functions from H^1 vanishing on the boundary and the space of bounded compactly supported functions, respectively.

Definition 2.6.1. *We say that a function $r = r(t,x,y)$ is a weak fundamental solution (heat kernel) of equation* (2.22) *if*

$$u(t,x) = \int_{\mathbb{R}^d} r(t,x,y)f(y)dy$$

is a solution to problem (2.23) *for all $t \in [0,T]$ and $f \in L_c^\infty(\mathbb{R}^d)$.*

Further on we shall also use the term fundamental solution for weak fundamental solution if this does not lead to confusions.

If (2.23) holds for every $T > 0$, then u is said to be a weak solution of the Cauchy problem

$$\partial_t u = \Lambda u, \ t > 0, \ \ u(0) = f.$$

A fundamental solution of (2.22) in the domain $[0,\infty) \times \mathbb{R}^d$ is defined similarly.

The fundamental solution r is said to enjoy the Gaussian upper and lower bounds if there are positive constants $\gamma, \bar\gamma, C_\gamma, C_{\bar\gamma}$ such that

$$C_{\bar\gamma} t^{-d/2} e^{-\frac{|x-y|^2}{4\bar\gamma t}} \le r(t,x,y) \le C_\gamma t^{-d/2} e^{-\frac{|x-y|^2}{4\gamma t}}, \quad 0 < t < T,\ x,y \in \mathbb{R}^d. \tag{2.24}$$

If (2.24) holds with $T = \infty$, then the Gaussian bounds are called *global (in time)*. Further on we use the notation $\Gamma(t,x) := (4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}}$ and $\Gamma_\alpha(t,x) := \Gamma(\alpha t, x)$ for $\alpha > 0$.

The validity of Gaussian upper bound on the heat kernel of equation (2.22) yields, in particular, a number of remarkable properties for the semigroup associated with the differential expression Λ. Below we comment on some of them.

Let r stand for the fundamental solution of (2.22) and the second estimate in (2.24) holds. One can readily see that r determines a C_0-semigroup on $L^1(\mathbb{R}^d)$ by

$$(U_\Lambda(t)f)(x) := \int_{\mathbb{R}^d} r(t,x,y)f(y)dy, \quad x \in \mathbb{R}^d,\ t \ge 0,\ f \in L^1(\mathbb{R}^d).$$

Similarly one can define a C_0-semigroup on $L^p(\mathbb{R}^d)$ for all $p \in (1,\infty)$. Next we observe that

$$\left|\int_{\mathbb{R}^d} r(t,x,y)f(y)dy\right| \le C_\gamma t^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4\gamma t}} |f(y)|dy \le Ct^{-d/2}\|f\|_1$$

for all $f \in L^1(\mathbb{R}^d)$. Hence, $\|U_\Lambda(t)\|_{1\to\infty} \le Ct^{-d/2}$. By the Fubini theorem we get

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} r(t,x,y)|f(y)|dydx \le C_\gamma t^{-d/2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4\gamma t}}|f(y)|dxdy \le C\|f\|_1.$$

Thus $\|U_\Lambda(t)\|_{1\to 1} \le C$. Using the Riesz-Thorin interpolation theorem (see e.g., [15, Th. 1.1.5]) we conclude that

$$\|\exp(t\widehat{\Lambda})f\|_p \le Ct^{-d/2p}\|f\|_1 \tag{2.25}$$

with some $C > 0$.

2.6.2 Some Remarks on Classes of Perturbations

Next we discuss some classes of potentials and drift coefficients.

Definition 2.6.2. *Let Ω be an open subset of $\mathbb{R}^d$ and $W \in L^1_{\mathrm{loc}}(\Omega)$. The potential W is said to belong to the Kato class $K_d(\Omega)$ if*

$$\lim_{\rho\to 0} M_d(W,\rho) := \lim_{\rho\to 0}\sup_{x\in\Omega} \int_{|x-y|\le\rho} \frac{|W(y)|}{|x-y|^{d-2}}dy = 0.$$

Analogously, the potential W belongs to the Kato class $K_{d+1}(\Omega)$ if

$$\lim_{\rho\to 0} M_{d+1}(W,\rho) := \lim_{\rho\to 0}\sup_{x\in\Omega} \int_{|x-y|\le\rho} \frac{|W(y)|}{|x-y|^{d-1}}dy = 0.$$

If there is a number $\rho > 0$ such that

$$M_d(W,\rho) < \eta$$

for some positive η, then we say that W is in the enlarged Kato class $\widehat{K}_d(\Omega)$. The enlarged Kato class $\widehat{K}_{d+1}(\Omega)$ is defined similarly.

It is clear that $K_m(\Omega) \subset \widehat{K}_m(\Omega)$, $m = d, d+1$. We set $K_m(\mathbb{R}^d) =: K_m$.

Next we make several observations regarding the Kato classes K_m and $\widehat{K}_m$, where $m = d$, $d+1$. Let $W \in K_d$. It is well-known (see e.g. [71]) that for all $\delta > 0$ there exists a $c(\delta) > 0$ such that

$$W \le \delta(-\Delta) + c(\delta)\,\mathrm{Id} \tag{2.26}$$

in the form sense. If $W \in \widehat{K}_d$ and $M_d(W,\rho)$ is sufficiently small for some $\rho > 0$, then there exist a $\delta \in (0,1)$ and $c(\delta) \ge 0$ such that the above estimate holds.

Let $W \in \widehat{K}_{d+1}$. Then there exists a constant $C = C(d) > 0$, independent of t, such that

$$\sup_{x\in\mathbb{R}^d} \int_0^t \int_{|x-y|\le\rho} \Gamma_\gamma(s, x-y) s^{-1/2} |W(y)| dy ds \le C\eta \tag{2.27}$$

Indeed, substitution of variables and the Fubini theorem imply that

$$\begin{aligned}
&\int_0^t \int_{|x-y|\le\rho} \Gamma_\gamma(s, x-y) s^{-1/2} |W(y)| dy ds \\
&= \frac{1}{4\gamma\pi^{d/2}} \int_{|x-y|\le\rho} \frac{|W(y)|}{|x-y|^{d-1}} \int_{\frac{|x-y|^2}{4\gamma t}}^{\infty} \tau^{(d-3)/2} e^{-\tau} d\tau dy \\
&\le \frac{1}{4\gamma\pi^{d/2}} \left(\int_0^\infty \tau^{(d-3)/2} e^{-\tau} d\tau \right) \int_{|x-y|\le\rho} \frac{|W(y)|}{|x-y|^{d-1}} dy \le C M_{d+1}(W, \rho).
\end{aligned}$$

A similar argument shows that there exists a constant $C = C(d) > 0$ such that

$$\sup_{x\in\mathbb{R}^d} \int_0^t \int_{|x-y|\le\rho} \Gamma_\gamma(s, x-y) |W(y)| dy ds \le C\eta,$$

whenever $W \in \widehat{K}_d$. The following result was in fact obtained in [53, Prop. 2.1] (see also [77, Appendix B], [67, Th. 2]). We give a complete proof of the statement in order to keep the exposition self-contained.

Proposition 2.6.3. *We assume that there exist positive numbers ε, ρ, t_0 such that*

$$\sup_{x\in\mathbb{R}^d} \int_0^{t_0} \int_{|x-y|\le\rho} \Gamma_\gamma(s, x-y) s^{-\delta} |W(y)| dy ds < \varepsilon,$$

with $0 \le \delta < 1$. Then there exists a $t \in (0, t_0)$ such that

$$\sup_{x\in\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \Gamma_\gamma(s, x-y) s^{-\delta} |W(y)| dy ds < \varepsilon.$$

Proof. Without loss of generality we may assume that $t_0 \le 1$. For $0 < t < t_0$ we have

$$\varepsilon > \sup_{x\in\mathbb{R}^d} \int\limits_0^t \int\limits_{|x-y|\le\rho/2} \Gamma_\gamma(s,x-y)|W(y)|dy\frac{ds}{s^\delta}$$
$$\ge \sup_{x\in\mathbb{R}^d} \int\limits_0^t (4\pi\gamma s)^{-d/2}e^{-\rho^2/16\gamma s} \int\limits_{|y|\le\rho/2} |W(x+y)|dy\frac{ds}{s^\delta}. \tag{2.28}$$

Using the Fubini theorem we estimate

$$\int\limits_0^t \int\limits_{|y|\ge\rho} (4\pi\gamma s)^{-d/2}e^{-|y|^2/4\gamma s}|W(x+y)|dy\frac{ds}{s^\delta}$$
$$\le C\int\limits_0^t (4\pi\gamma s)^{-d/2}e^{-\rho^2/8\gamma s} \int\limits_{|y|\ge\rho}\int\limits_{|z-y|\le\rho/3} e^{-|y|^2/8\gamma}|W(x+y)|dzdy\frac{ds}{s^\delta}$$
$$\le C\int\limits_0^t (4\pi\gamma s)^{-d/2}e^{-\rho^2/8\gamma s} \int\limits_{|z|\ge 2\rho/3}\int\limits_{|y|\le\rho/3} e^{-|y+z|^2/8\gamma}|W(x+y+z)|dydz\frac{ds}{s^\delta}$$
$$\le C\int\limits_0^t (4\pi\gamma s)^{-d/2}e^{-\rho^2/8\gamma s} \int\limits_{|z|\ge 2\rho/3} e^{-|z|^2/32\gamma}\int\limits_{|y|\le\rho/3} |W(x+y+z)|dydz\frac{ds}{s^\delta}$$
$$\le Ce^{-\rho^2/16\gamma t}\int\limits_{\mathbb{R}^d} e^{-|z|^2/32\gamma}\left(\int\limits_0^t (4\pi\gamma s)^{-d/2}e^{-\rho^2/16\gamma s}\int\limits_{|y|\le\rho/3} |W(x+y+z)|dy\frac{ds}{s^\delta}\right)dz$$
$$\le \hat{C}e^{-\rho^2/16\gamma t}\varepsilon, \tag{2.29}$$

where the last inequality follows from (2.28). Making use of (2.29) we conclude that

$$\sup_{x\in\mathbb{R}^d}\int\limits_0^t\int\limits_{\mathbb{R}^d} \Gamma_\gamma(s,x-y)s^{-\delta}|W(y)|dyds$$
$$< \sup_{x\in\mathbb{R}^d}\int\limits_0^{t_0}\int\limits_{|x-y|\le\rho} \Gamma_\gamma(s,x-y)s^{-\delta}|W(y)|dyds + \hat{C}e^{-\rho^2/16\gamma t}\varepsilon.$$

Taking $t>0$ sufficiently small we complete the proof. □

Let $T\in(0,\infty)$. For positive γ and h and measurable $w:(0,T)\times\mathbb{R}^d\to\mathbb{R}$ we set

$$M_{d+1}(w,h,\gamma) := \sup_{x,s}\int\limits_s^{(s+h)\wedge T}\int\limits_{\mathbb{R}^d}\Gamma_\gamma(t-s,x-y)(t-s)^{-1/2}|w(t,y)|dydt.$$

By $P\widehat{K}^{\gamma}_{d+1}$ we denote the set of all w such that $M_{d+1}(w,h,\gamma) < \infty$ for some (and hence for all) $h > 0$ (cf. [80]). Obviously, $0 < \gamma_1 \le \gamma_2$ yields $P\widehat{K}^{\gamma_2}_{d+1} \subset P\widehat{K}^{\gamma_1}_{d+1}$.

Repeating an argument from [78, Prop. 3.2] we see that $P\widehat{K}^{\gamma}_{d+1}$ is a Banach space with the norm given by $\|w\| := M_{d+1}(w,h,\gamma)$ for some $h > 0$. It is readily seen that every measurable function $\phi : (0,T) \times \mathbb{R}^d \to \mathbb{R}$, satisfying the estimate

$$|\phi(t,y)| \le \int_{\mathbb{R}^d} (t-s)^{-1/2}\Gamma_\gamma(t-s,x-y)|f(x)|dx$$

with some $0 < s < t$ and $f \in L^1(\mathbb{R}^d)$, defines a linear bounded functional on $P\widehat{K}^{\gamma}_{d+1}$ by

$$\Phi(w) := \int_0^T \int_{\mathbb{R}^d} \phi(t,y)w(t,y)dydt.$$

Now let $W \in \widehat{K}_{d+1}$. It follows from (2.27) and Proposition 2.6.3 that $W \in P\widehat{K}^{\gamma}_{d+1}$ and there exist constants $C > 0$ and $t > 0$ such that

$$M_{d+1}(W,t,\gamma) = \sup_{x\in\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \Gamma_\gamma(s,x-y)s^{-1/2}|W(y)|dyds < C\eta. \tag{2.30}$$

Similarly,

$$M_d(W,t,\gamma) := \sup_{x\in\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \Gamma_\gamma(s,x-y)|W(y)|dyds < C\eta, \tag{2.31}$$

whenever $W \in \widehat{K}_d$.

Remark. *It can be proved by the same means that*

$$\lim_{t\to 0} M_{d+1}(W,t,\gamma) = 0, \quad \textit{if } W \in K_{d+1},$$

and

$$\lim_{t\to 0} M_d(W,t,\gamma) = 0, \quad \textit{if } W \in K_d.$$

Let $p = p(t,x,y)$ stand for the heat kernel of the equation

$$\partial_t u = \nabla \cdot a \cdot \nabla u,$$

with a uniformly elliptic matrix a. The function p is known to enjoy the two-sided Gaussian estimates, so the differential expression $\nabla \cdot a \cdot \nabla$ gives rise to a

C_0-semigroup on $L^1(\mathbb{R}^d)$. By $\mathcal{A}$ we denote its generator. We assume, in addition, that the entries of a are uniformly Hölder continuous. Then, following a method described in [44, Ch.IV, Sections 11-13], one can prove that

$$|\nabla_x p(t,x,y)| \le c_\beta t^{-\frac{1}{2}} \Gamma_\beta(t, x-y), \ t > 0, \tag{2.32}$$

with some positive β and c_β.

Let $|b| \in \widehat{K}_{d+1}$ and $V \in \widehat{K}_d$. We define an operator $\mathcal{B}$ on the domain

$$\mathcal{D}(\mathcal{B}) := \{u \in L^1(\mathbb{R}^d) \,|\, \nabla u \in L^1_{\mathrm{loc}}(\mathbb{R}^d),\, b \cdot \nabla u \in L^1(\mathbb{R}^d)\}$$

by $\mathcal{B}u := -b \cdot \nabla u$, $u \in \mathcal{D}(\mathcal{B})$. Similarly, an operator $\mathcal{V}$ is defined by $\mathcal{V}u := -Vu$, $u \in \mathcal{D}(\mathcal{V})$, where

$$\mathcal{D}(\mathcal{V}) := \{u \in L^1(\mathbb{R}^d) \,|\, Vu \in L^1(\mathbb{R}^d)\}.$$

It follows from (2.32), the Fubini theorem, (2.30) and (2.31) that

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(|b(x) \cdot \nabla_x p(s,x,y)| + |V(x)| p(s,x,y)\right) |f(y)| dy dx ds \\
&\le C_\beta \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Big(|b(x)| \Gamma_\beta(s, x-y) s^{-1/2} + |V(x)| \Gamma_\beta(s, x-y)\Big) |f(y)| dx dy ds \\
&= C_\beta \int_{\mathbb{R}^d} |f(y)| \left(\int_0^t \int_{\mathbb{R}^d} \Big(|b(x)| \Gamma_\beta(s, x-y) s^{-1/2} + |V(x)| \Gamma_\beta(s, x-y)\Big) dx ds \right) dy \\
&\le C_\beta (M_{d+1}(|b|, t, \beta) + M_d(V, t, \beta)) \|f\|_1 \le 2 C_\beta C \eta \|f\|_1,
\end{aligned}$$

for all $f \in \mathcal{D}(\mathcal{A})$ and $t > 0$. Thus $2C_\beta C\eta < 1$, provided η is sufficiently small. Hence, the Miyadera theorem implies that the operator $H = \mathcal{A} + \mathcal{B} + \mathcal{V}$, with the domain $\mathcal{D}(H) = \mathcal{D}(\mathcal{A})$, is the generator of a C_0-semigroup on $L^1(\mathbb{R}^d)$, i.e. there are numbers $\omega \ge 0$ and $M \ge 1$ such that

$$\|\exp(tH) f\|_1 \le M e^{\omega t} \|f\|_1$$

for all $f \in L^1(\mathbb{R}^d)$.

2.6.3 Elliptic Equations

Next we discuss elliptic equations of the type

$$\Lambda v = 0 \ \text{ in } \ \Omega, \tag{2.33}$$

where Ω is an open subset of $\mathbb{R}^d$.

A distribution G is called a fundamental solution of equation (2.33) with $\Omega = \mathbb{R}^d$ if $\Lambda G = \delta$, where δ stands for the Dirac δ-function.

Further on we shall make extensive use of the *maximum principle* and the *Harnack inequality,* related to equation (2.33). We begin by formulating the classical results.

Proposition 2.6.1. *([74, Th. I.10.3].) Let Ω be an open connected subset (a domain) of $\mathbb{R}^d$ and K a compact subset of Ω. There exists a constant $C_K > 0$ such that for every positive solution v of the equation $\Delta v = 0$ the following estimate holds:*

$$\sup_{x \in K} v(x) \leq C_K \inf_{x \in K} v(x).$$

Next we formulate the classical maximum principle.

Proposition 2.6.2. *([74, Th. I.10.2].) Let Ω be a domain in $\mathbb{R}^d$. Let v be a solution of the equation $\Delta v = 0$. Then either $v \equiv const$, or*

$$|v(x)| < \sup_{\Omega} |v| \quad \textit{for all} \quad x \in \Omega.$$

This result is generalised in the next subsection (see Proposition 2.6.4).

The Harnack inequality for positive weak solutions of (2.33) in bounded domains was established in [42].

Proposition 2.6.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. We assume that the matrix a is uniformly elliptic and the potentials $|b|^2$, $V \in K_d(\Omega)$. Let v be a positive weak solution of (2.33). Then there exist constants $r_0 > 0$ and $C > 0$ such that we have*

$$\sup_{B_r} v \leq C \inf_{B_r} v.$$

for every $r \in (0, r_0)$ with $B_{4r} \subset \Omega$.

Remark. *Combining results obtained in [80, 82] one can prove the Harnack inequality under assumptions that a is a uniformly elliptic matrix with Hölder continuous entries, $|b| \in K_{d+1}(\Omega)$ and $V \in K_d(\Omega)$.*

2.6.4 Semi-linear Inequalities. Existence and Non-existence of Positive Solutions

Let Ω be a domain in $\mathbb{R}^d$ and $F : \Omega \times [0, \infty) \to [0, \infty)$ be measurable. We assume, in addition, that $|b|^2, V \in L^1_{\mathrm{loc}}(\Omega)$. A measurable function u is called a weak super-solution (solution) of the equation (inequality)

$$\Lambda u + F = (\leq) 0,$$

if $u \in H^1_{\mathrm{loc}}(\Omega) \cap L^2_{\mathrm{loc}}(\Omega, (|b|^2 + |V|)dx)$ and

$$\int_{\mathbb{R}^d} \big(\nabla u(x) \cdot a(x) \cdot \nabla \phi(x) + b(x) \cdot \nabla u(x) \phi(x) - V(x) u(x) \phi(x) - F(x, u(x)) \phi(x)\big) dx \geq 0,$$

for all $0 \leq \phi \in H^1_{\mathrm{c}}(\Omega) \cap L^2(\Omega, (|b|^2 + |V|)dx)$, where H^1_{c} stands for the space of compactly supported functions in H^1.

In the future we shall make use of the following version of the maximum principle.

Proposition 2.6.4. *Let $R > 0$ and u be a super-solution of*

$$\nabla \cdot a \cdot \nabla u - b \cdot \nabla u - Vu = 0 \quad in \quad B^c_R.$$

We assume that $|b|^2 + |V| \leq \alpha(-\nabla \cdot a \cdot \nabla)$ in the form sense for some $\alpha \in (0, 1)$. We also assume that $u^- \restriction_{\partial B_R} = 0$ in the sense of H^1 and $u^- \in L^2(B^c_R, |x|^{-2}dx)$. Then $u \geq 0$.

Proof. The proof can be easily carried out by multiplying the equation by the test function $\theta^2 u^-$, where θ is defined by

$$\theta_\rho(x) = \begin{cases} \rho(|x| - R) - 1, & x \in A_{R+\frac{2}{\rho}, R+\frac{1}{\rho}}; \\ 1, & x \in A_{\rho, R+\frac{2}{\rho}}; \\ 2 - \frac{|x|}{\rho}, & x \in A_{2\rho, \rho}, \end{cases}$$

with $\rho > R + 2$. For details see e.g. [36]. □

Further on we need an existence result for the problem

$$\begin{aligned} &\nabla \cdot a \cdot \nabla v - b \cdot \nabla v - Vv = 0 \ \text{ in } \ B^c_R, \\ &v \restriction_{\partial B_R} = \psi \restriction_{\partial B_R} \geq 0, \end{aligned} \tag{2.34}$$

where $R > 0$ and $\psi \in H^1_c(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Let

$$\mathcal{H}_R := \left\{ v \in H^1_{\mathrm{loc}}(B^c_R) \,\middle|\, \int_{B^c_R} \left(\nabla v \cdot a \cdot \nabla v + \frac{v^2}{|x|^2} \right) dx < \infty \right\}.$$

For $0 < R_1 < R_2$ we set $A_{R_2,R_1} := B_{R_2} \setminus \overline{B_{R_1}}$.

Proposition 2.6.5. *We assume that $|b|^2 + |V| \leq \alpha(-\nabla \cdot a \cdot \nabla)$ in the form sense for some $\alpha \in (0,1)$. Then for each positive $\psi \in H^1_c(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ there exists a unique solution $v \in \mathcal{H}_R$ to (2.34). Furthermore, $v \geq 0$.*

Proof. Let $\rho > R$ be such that $\psi \in H^1_c(B_\rho)$. It follows from the Lax-Milgram theorem (see e.g., [21, Th. IV.1.1]) that the problem

$$\begin{aligned} &\nabla \cdot a \cdot \nabla v - b \cdot \nabla v - Vv = 0 \quad \text{in} \quad A_{\rho,R}, \\ &v \restriction_{\partial B_R} = \psi \restriction_{\partial B_R} \geq 0, \; v \restriction_{\partial B_\rho} = 0, \end{aligned}$$

has a unique solution which is denoted by v_ρ. Taking $v_\rho - \psi$ as a test function, integrating by parts and using the conditions of the proposition we derive the estimate

$$\|\nabla v_\rho\|_2^2 \leq C\Big(\|\nabla \psi\|_2^2 + \|b\psi\|_2^2 + \|V\psi\|_2^2 \Big),$$

with C independent of ρ. Therefore there exists a subsequence $(v_{\rho_n})_{n\in\mathbb{N}}$, $\rho_n \to \infty$, which is weakly convergent in $\mathcal{H}_R$ and the limit v is easily seen to be a solution to (2.34). It follows from Proposition 2.6.4 that the solution v is unique. $\square$

In Chapter 5 we are concerned with the existence and non-existence of positive super-solutions of the equation

$$\nabla \cdot a \cdot \nabla u - b \cdot \nabla u - Vu + u^p = 0$$

in $\Omega := K^c$, where $p > 1$ and K is a compact subset of $\mathbb{R}^d$. Our main result (Theorem 5.2.1) is stated and proved in section 5.2. In order to illustrate the method, used for establishing this result, we give a detailed proof of the statement in the case $a = \mathrm{Id}$ and $b, V = 0$.

Thus until the end of this section we study the equation

$$\Delta u + u^p = 0 \quad \text{in} \quad K^c. \tag{2.35}$$

Let $p_0 := \dfrac{d}{d-2}$. The relevant result reads as follows.

Theorem 2.6.4. *Let $p > 1$. Then the following assertions are equivalent:*

(i) $p \leq p_0$;

(ii) equation (2.35) *does not have non-trivial positive weak super-solutions.*

The number p_0 *is called the critical exponent for equation* (2.35).

In order to prove Theorem 2.6.4 we need several auxiliary results. Lemma 2.6.5 is the main tool for proving non-existence of positive weak super-solutions.

Lemma 2.6.5. *Let* Ω *be an open bounded subset of* $\mathbb{R}^d$ *and* $0 \leq W \in L^\infty_{\mathrm{loc}}(\Omega)$. *Let* u *be a positive solution of the inequality*

$$\Delta u + Wu \leq 0.$$

Then there exists a $\lambda_0 > 0$ *such that*

$$(W > \lambda_0) \Longrightarrow (u \equiv 0).$$

Proof. This lemma is a particular case of a more general result proved in Chapter 5 (see Lemma 5.2.3). □

Without loss of generality we can assume that $K \subset B_1$. In the next lemma we establish an a priori estimate for a non-trivial positive weak solution of (2.35).

Lemma 2.6.6. *Let* u *be a positive weak super-solution of* (2.35). *If* u *is non-trivial, then there exist constants* $c_0 > 0$ *and* $R_0 > 1$ *such that*

$$u(x) \geq c_0 |x|^{2-d} \quad \text{in} \quad B^c_{R_0}.$$

Proof. Let u be a solution of (2.35). One can easily see that u is a solution of the inequality

$$\Delta u \leq 0 \quad \text{in} \quad B^c_1.$$

Since $u \in H^1_{\mathrm{loc}}(B^c_1)$ one can readily see that $u \in H^1(A_{6,2})$. So we can consider the problem

$$\begin{aligned} &\Delta v = 0, \\ &v \restriction_{\partial B_2} = u \restriction_{\partial B_2} \geq 0, \quad v \restriction_{\partial B_6} = u \restriction_{\partial B_6} \geq 0. \end{aligned}$$

The maximum principle (Proposition 2.6.4) implies that $v \geq 0$ on $A_{6,2}$. Making use of the classical Harnack inequality (Proposition 2.6.1) we conclude that there is a constant $\bar{c} > 0$ such that $v \geq \bar{c}$ on $A_{5,3}$. Next we set $w := u - v$ in $A_{5,3}$, and observe that w satisfies the conditions of Proposition 2.6.4. Hence, $u \geq v \geq \bar{c} > 0$ in $A_{5,3}$. In particular, $u \geq \bar{c}$ on ∂B_4.

It is obvious that one can find a constant $c_0 > 0$ such that $u(x) \geq c_0|x|^{2-d}$ on ∂B_4. Setting $w(x) := u(x) - c_0|x|^{2-d}$, $x \in B_4^c$, and noting that $w^- \leq c_0|x|^{2-d}$ we see that the conditions of Proposition 2.6.4 are fulfilled again. This implies that $u(x) \geq c_0|x|^{2-d}$ in B_4^c. □

Using Lemmas 2.6.5 and 2.6.6 we can prove the implication (i)⇒(ii) in Theorem 2.6.4 in the case $p < p_0$.

Proof of Theorem 2.6.4. The case $p < p_0$. Let u be a positive weak super-solution of (2.35). We are going to show that $u \equiv 0$. For contradiction we assume that this is not the case. The rest of the proof is divided into two steps.

Step 1. It is easily seen that u is a solution of the inequality

$$\Delta u \leq 0 \text{ in } B_1^c.$$

By Lemma 2.6.6 there are constants $\hat{c} > 0$ and $\delta > 0$ such that $u^{p-1}(x) \geq \hat{c}|x|^{-2+\delta}$ in $B_{R_0}^c$. One can readily see that u is a solution of the inequality

$$\Delta u + \hat{c}|x|^{-2+\delta} u \leq 0 \text{ in } A_{2\rho,\rho}$$

for every $\rho > R_0$. Writing $x = \rho x'$, $x' \in [1, 2]$, we get

$$\frac{1}{\rho^2}\Delta_{x'} u + \frac{\hat{c}\rho^\delta |x'|^\delta}{\rho^2|x'|^2} u \leq 0,$$

or,

$$\Delta_{x'} u + \hat{c}\rho^\delta |x'|^{-2+\delta} u \leq 0 \text{ in } A_{2,1}.$$

Finally, making use of Lemma 2.6.5, we conclude that $u \equiv 0$ in $B_{\rho_0}^c$, where ρ_0 is determined by the relation $\rho_0^\delta = \lambda_0$ and λ_0 is a constant from Lemma 2.6.5. Hence, $u \equiv 0$ in $B_{R_0}^c$.

Step 2. In order to complete the proof we show that $u \equiv 0$ in $B_{R_0} \setminus K$. Suppose $u > 0$ in a small ball B in $B_{R_0} \setminus K$. Let Ω be a bounded domain, such that $\Omega \cap B_{\rho_0}^c \neq \emptyset$ and $\partial\Omega \cap B \neq \emptyset$. We study the Dirichlet problem

$$\Delta v = 0 \text{ in } \Omega,$$
$$v \restriction_{\partial\Omega} = u \restriction_{\partial\Omega} \geq 0.$$

By the maximum principle we infer that $v \geq 0$ in Ω. The Harnack inequality implies that $v > 0$ in $\Omega' \subset \Omega \cap B^c_{\rho_0} \neq \emptyset$. However, Proposition 2.6.4 yields $u \geq v > 0$ in $B^c_{R_0}$ which contradicts Step 1. $\square$

The case of the critical exponent appears to be more delicate. In order to complete the proof of the implication (i)⇒(ii) in Theorem 2.6.4 we need an estimate of the solution to the problem

$$\begin{aligned} &\Delta v + \nu |x|^{-2} v = 0 \ \text{ in } \ B^c_R, \\ &v \restriction_{\partial B_R} = \mu > 0, \end{aligned} \tag{2.36}$$

with some $\nu > 0$ and $R > 1$. The following result holds.

Lemma 2.6.7. *For all sufficiently small $\nu > 0$ there exist a unique solution to problem (2.36) and numbers $R_1 > 1$ and $C_0 > 0$ such that*

$$v(x) \geq C_0 |x|^{2-d} \log |x| \quad in \quad B^c_{R_1}.$$

Proof. It follows from the Hardy inequality that

$$\nu \langle |x|^{-2} \varphi^2 \rangle \leq \nu c_H \|\varphi\|_2^2.$$

Hence, by Proposition 2.6.5 there exists a unique positive solution $v \in \mathcal{H}_1$ to (2.36), provided $\nu c_H < 1$.

Next we establish an estimate of the Caccioppoli type for the solution v. Namely, for every $\rho > 2R$ we have

$$\int_{A_{2\rho,\rho}} |\nabla v(x)|^2 dx \leq \hat{C} m_\rho^2 \rho^{d-2}, \tag{2.37}$$

where $m_\rho := \inf\limits_{|x|=\rho} v(x)$ and $\hat{C}$ is independent of ρ.

A simple rescaling argument and the Harnack inequality imply that $c m_\rho \leq v \leq C m_\rho$ in $A_{2\rho,\rho}$, with c, C independent of ρ. Indeed, let v stand for a positive solution of the equation

$$\Delta v + \nu |x|^{-2} v = 0 \ \text{ in } \ A_{3\rho,\frac{\rho}{2}}.$$

We set $x := \rho x'$, $1 \leq x' \leq 2$, and obtain the equation

$$\Delta_{x'} v(\rho \cdot) + \nu |x'|^{-2} v(\rho \cdot) = 0 \ \text{ in } \ A_{3,1/2}.$$

By the Harnack inequality there are constants c, $C > 0$ such that

$$c \inf_{|x'|=1} v(\rho x') \le v(\rho x') \le C \sup_{|x'|=1} v(\rho x') \quad \text{in } A_{2,1}.$$

Let $\rho > 2R$. Let $\varphi \in C_0(\mathbb{R}^d)$ such that $0 \le \varphi \le 1$, $\varphi = 1$ in $A_{2\rho,\rho}$ and $\operatorname{supp}\varphi \subset A_{\frac{5\rho}{2},\frac{\rho}{2}}$. Hence, $\varphi^2 v$ can be taken as a test function. Integrating by parts we get

$$0 = \langle \Delta v + c|x|^{-2}v,\, \varphi^2 v\rangle = -\langle \nabla v,\, \nabla(\varphi^2 v)\rangle + c\langle |x|^{-2}(fv)^2\rangle.$$

We note that $\nabla v \cdot \nabla(\varphi^2 v) = v^2|\nabla\varphi|^2 - |\nabla(\varphi v)|^2$ and obtain the equality

$$\|\nabla(\varphi v)\|_2^2 = c\langle |x|^{-2}(\varphi v)^2\rangle + \|v|\nabla\varphi|\,\|_2^2.$$

Using the definition of φ and the Harnack inequality we conclude that

$$\begin{aligned}\int_{A_{2\rho,\rho}} |\nabla v(x)|^2 dx \le & \int_{A_{\rho,\frac{\rho}{2}} \cup A_{\frac{5\rho}{2},2\rho}} v(x)^2 |\nabla\varphi(x)|^2 dx + c \int_{A_{\frac{5\rho}{2},\frac{\rho}{2}}} \left(\frac{v(x)\varphi(x)}{|x|}\right)^2 dx \\ \le & c_1 m_\rho^2 \rho^{-2} \int_{[\frac{\rho}{2},\rho]\cup[2\rho,\frac{5\rho}{2}]} r^{d-1} dr + c_2 m_\rho^2 \int_{[\frac{\rho}{2},\frac{5\rho}{2}]} r^{d-3} dr \le \hat{c}\, m_\rho^2 \rho^{d-2}. \end{aligned} \tag{2.38}$$

Hence (2.37) is proved.

Now let $\phi \in C_0(\mathbb{R}^d)$ such that $0 \le \phi \le 1$, $\phi = 1$ in $A_{\rho,2R}$ and $\operatorname{supp}\phi \subset A_{2\rho,3/2}$. Then ϕ is a test function. Integration by parts yields

$$c\langle |x|^{-2}\phi v\rangle = \langle \nabla\phi,\, \nabla v\rangle. \tag{2.39}$$

Making use of Lemma 2.35 we estimate the left-hand side of (2.39) below by

$$c \int_{A_{\rho,2R}} |x|^{-2} v(x) dx \ge \hat{c} \int_{A_{\rho,2R}} |x|^{-d} dx = \bar{c}\,(\log\rho - \log 2).$$

Recalling that $v \in \mathcal{H}_1$, using the Schwarz inequality and employing (2.38) we estimate the right-hand side of (2.39) above by

$$\begin{aligned} & c_1 \int_{A_{2R,\frac{3R}{2}}} |\nabla v(x)| dx + c_2 \rho^{-1} \int_{A_{2\rho,\rho}} |\nabla v(x)| dx \\ \le & \bar{c}_1 + \hat{c}_1 \Big(\int_{A_{2\rho,\rho}} |\nabla v(x)|^2 dx \Big)^{1/2} \rho^{d/2} \le \bar{c}_1 + \bar{c}_2 m_\rho \rho^{d-2}. \end{aligned}$$

Substituting the derived inequalities into (2.39) we get

$$\bar{c}_0 \log \rho \le \bar{c}_1 + \bar{c}_2 m_\rho \rho^{d-2}.$$

Therefore there exist constants R_1 and $C > 0$ such that

$$m_\rho \ge C\rho^{2-d} \log \rho \quad \text{for all} \quad \rho > R_0.$$

Finally, we employ the Harnack inequality and complete the proof. □

Now we are ready to treat the case of the critical exponent in Theorem 2.6.4.

Proof of Theorem 2.6.4. The case $p = p_0$. Let u be a positive weak solution of (2.35). Using Lemma 2.6.6 and observing that $p_0 - 1 = \frac{2}{d-2}$ we conclude that u is a solution of the inequality

$$\Delta u + c_0^{\frac{2}{d-2}} |x|^{-2} u \le 0 \quad \text{in} \quad B_{R_0}^c.$$

We set $\kappa := \nu \wedge c_0^{\frac{2}{d-2}}$, where ν is the same as in Lemma 2.6.7. Then u is a solution to the problem

$$\begin{aligned} &\Delta u + \kappa |x|^{-2} u \le 0 \quad \text{in} \quad B_{R_0}^c, \\ &u \restriction_{\partial B_{R_0}} \ge 0. \end{aligned}$$

By v we denote the solution to the problem

$$\begin{aligned} &\Delta v + \kappa |x|^{-2} v = 0 \quad \text{in} \quad B_{R_0}^c, \\ &v \restriction_{\partial B_{R_0}} = u \restriction_{\partial B_{R_0}} \ge 0. \end{aligned}$$

We set $w := u - v$ and observe that w^- satisfies the conditions of Proposition 2.6.4. Hence, $w \ge 0$ in $B_{R_0}^c$. Lemma 2.6.7 implies that $u(x) \ge v(x) \ge C_0 |x|^{2-d} \log |x|$, $x \in B_{R_1}^c$. It is readily seen that u is a solution of the inequality

$$\Delta u + C|x|^{-2} \log^{\frac{2}{d-2}} |x| u \le 0 \quad \text{in} \quad A_{2\rho,\rho}$$

for all $\rho > R_1$. Similar to the proof in the case $p < p_0$, a rescaling argument and using Lemma 2.6.5 imply that $u \equiv 0$ in $B_{\rho_0}^c$, where ρ_0 is determined by the relation $\log^{\frac{2}{d-2}} \rho_0 = \lambda_0$. Thus $u \equiv 0$ in $B_{R_0}^c$. The rest of the proof is the same as in the case $p < p_0$. □

Proposition 2.6.8 below completes the proof of Theorem 2.6.4.

Proposition 2.6.8. *If $p > p_0$, then one can find a number $\mu > 0$ such that there exists a positive solution $u \in \mathcal{H}_1$ to the problem*

$$\begin{aligned} &\Delta u + u^p = 0 \quad in \quad B_1^c, \\ &u \restriction_{\partial B_1} = \mu. \end{aligned}$$

The proof of Proposition 2.6.8 is based on the celebrated Schauder fixed-point theorem.

Theorem 2.6.9. *(e.g. [28, Cor. 10.2]). Let X be a Banach space. Let $A \subset X$ be closed and convex. Let $F : X \to X$ be a continuous mapping such that $F(A) \subset A$ and $F(A)$ is compact. Then there exists an element $a \in A$ such that $F(a) = a$.*

Proof of Proposition 2.6.8. Let $p > p_0$. For $R > 1$ we set

$$S_R := \{f \in L^2(A_{R,1}) \,|\, 0 \le f(x) \le c_0|x|^{2-d}\},$$

where the number $c_0 > 0$ is to be determined later. Let $\mu > 0$. We study the problem

$$\begin{aligned} &\Delta u + f^{p-1}u \le 0 \quad \text{in} \quad A_{R,1}, \\ &u \restriction_{\partial B_1} = \mu, \; u \restriction_{\partial B_R} = 0, \end{aligned} \tag{2.40}$$

where $f \in S_R$. A direct computation shows that $f^{p-1}(x) \le c_0^{p-1}|x|^{-2-\varepsilon}$, $x \in A_{R,1}$, with $\varepsilon := (d-2)(p-p_0) > 0$. The Hardy inequality implies that

$$\langle c_0^{p-1}|x|^{-2-\varepsilon}\varphi^2\rangle \le c_H c_0^{p-1}\|\nabla\varphi\|_2^2.$$

Hence, by Proposition 2.6.5, there exists a unique positive solution u_R to (2.40), provided c_0 is sufficiently small.

Let the mapping $F : S_R \to L^2(A_{R,1})$ be defined by $F(f) = u_R$, $f \in S_R$. We note that the set S_R is a closed, convex subset of $L^2(A_{R,1})$. Our goal is to show that $F(S_R) \subset S_R$ and $F(S_R)$ is compact in $L^2(A_{R,1})$.

Let $\overline{W} := c|x|^{-2-\varepsilon}\mathbb{1}_{B_1^c}$, with $c > 0$ to be chosen later. By $G_\varepsilon = G_\varepsilon(x,y)$ we denote the fundamental solution of the equation

$$\Delta v + \overline{W}v = 0.$$

It follows from [83] that there are positive numbers C_1 and C_2 such that

$$C_1|x-y|^{2-d} \le G_\varepsilon(x,y) \le C_2|x-y|^{2-d} \quad \text{for all} \quad x \ne y. \tag{2.41}$$

Let $v(x) := \bar{c}G_\varepsilon(x,0)$ with $\bar{c} > 0$. Let $R > 0$ be the same as in (2.40) and set $w_R := v - u_R$ in $A_{R,1}$. A straightforward computation yields

$$\begin{aligned}
&\Delta w_R + c|x|^{-2-\varepsilon} w_R = (f^{p-1} - c|x|^{-2-\varepsilon}) u_R \le 0 \ \text{ if } \ c_0^{p-1} \le c,\\
&w_R \restriction_{\partial B_1} = \bar{c}G_\varepsilon(1,0) - \mu \ge \bar{c}C_1 - \mu > 0 \ \text{ if } \ \mu < \bar{c}C_1,\\
&w_R \restriction_{\partial B_R} = \bar{c}G_\varepsilon(R,0) \ge \bar{c}C_1 R^{2-d} > 0.
\end{aligned}$$

We also note that $w_R^- \le u_R$, so $w_R^- \in L^2(A_{R,1}, |x|^{-2}dx)$. Choosing c to be small enough and making use of Proposition 2.6.4 we infer that $w_R \ge 0$. Hence, by (2.41), we have $u_R \le v \le \bar{c}C_2|x|^{2-d}$. We take $c \le C_2^{-1}c_0$ and conclude that $u_R \in S_R$. We shall see below that $u_R \in H^1(A_{R,1})$. Hence, $F(S_R) \subset H^1(A_{R,1}) \cap S_R$.

Next we observe that the mapping F is continuous. Indeed, let $f, (f_n)_{n\in\mathbb{N}} \subset S_R$ be such that $f_n \to f$ in $L^2(A_{R,1})$. By u_n, $n \in \mathbb{N}$, we denote the solution to the problem (2.40) with f replaced by f_n. It is easy to see that $w_n := u - u_n$ is a solution to the problem

$$\begin{aligned}
&Lw_n + f^{p-1}w_n + u_n(f^{p-1} - f_n^{p-1}) = 0,\\
&w_n \restriction_{\partial B_1} = w_n \restriction_{\partial B_R} = 0.
\end{aligned}$$

We note that w_n can be taken as a test function. Integrating over $A_{R,1}$, and applying the Hardy inequality we conclude that

$$\|\nabla w_n\|_2^2 \le C\big|\langle u_n w_n,\ f^{p-1} - f_n^{p-1}\rangle\big|.$$

It follows from the dominated convergence theorem that $\|\nabla w_n\|_2 \to 0$. Hence, by the Hausdorff inequality $w_n \to 0$ in $L^2(A_{R,1})$.

Now Theorem 2.6.9 implies that the mapping F has a fixed point $\bar{u}_R \in S_R$, i.e. $\bar{u}_R$ is the solution to the problem

$$\begin{aligned}
&\Delta u + u^p \le 0 \ \text{ in } \ A_{R,1},\\
&u \restriction_{\partial B_1} = \mu,\ u \restriction_{\partial B_R} = 0,
\end{aligned}$$

provided $\mu < c_0 < (c_H)^{\frac{1}{p-1}}$.

Further on we denote $\bar{u}_R$ by u_R. Let $\phi \in C_0^1(\mathbb{R}^d)$, $\phi \restriction_{\partial B_1} = \mu$ and $\operatorname{supp}\phi \subset A_{\frac{3}{2},\frac{1}{2}}$. Since $u_R - \phi \restriction_{\partial B_1} = u_R - \phi \restriction_{\partial B_R} = 0$ we can take $u_R - \phi$ as a test function. We have

$$0 = \langle \Delta u_R + u_R^p,\ u_R - \phi\rangle = -\|\nabla u_R\|_2^2 + \langle \nabla u_R,\ \nabla\phi\rangle + \langle u_R^{p+1}\rangle - \langle u_R^p \phi\rangle.$$

Using the Cauchy inequality we obtain the estimate

$$\|\nabla u_R\|_2^2 \leq \|\nabla \phi\|_2^2 + \langle u_R^{p+1} \rangle \leq C.$$

In particular, the last estimate yields $u_R \in H^1(A_{R,1})$ for all $R > 1$. A direct computation shows that

$$\left\| \frac{u_R}{r} \right\|_2^2 \leq c_0^2 c_d \int_1^R r^{2-2d} r^{d-1} dr \leq C.$$

Therefore the set $(u_R)_{R>1}$ is uniformly bounded in $\mathcal{H}_1$. So there exist a sequence $(u_{R_n})_{n \in \mathbb{N}}$, $R_n \to \infty$, and a function $u \in \mathcal{H}_1$ such that $u_{R_n} \to u$ as $n \to \infty$ weakly in $\mathcal{H}_1$. Hence u is a weak solution to problem (2.35) for all sufficiently small $\mu > 0$. $\square$

Chapter 3

L^p-Uniqueness for Perturbed Dirichlet Operators

In this chapter we are concerned with the uniqueness problem in $L^p := L^p(\mathbb{R}^d, \rho dx)$ ($\rho > 0$ a.e. and $\rho \in L^1_{\text{loc}}(\mathbb{R}^d, dx)$) for the second order operators associated with the differential expression

$$\mathcal{H}^{(b,q)} = -\sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2} - \sum_{k=1}^{d} \beta_k(x) \frac{\partial}{\partial x_k} + \sum_{k=1}^{d} b_k(x) \frac{\partial}{\partial x_k} + q(x), \tag{3.1}$$

where $\beta := (\beta_1, ..., \beta_d)$, $\beta_k = \dfrac{1}{\rho} \dfrac{\partial \rho}{\partial x_k}$, $k = 1, \ldots, d$ is the logarithmic derivative of the measure ρdx, $b = (b_1, \ldots, b_d) : \mathbb{R}^d \to \mathbb{R}^d$ and $q : \mathbb{R}^d \to \mathbb{C}$ are measurable (see section 2.4 for the definition and some classical results on L^2-uniqueness for Schrödinger operators).

Let $\mathcal{L}$ stand for the operator in L^2 associated with the closure of the sesquilinear form

$$\mathcal{E}(u, v) = \langle \nabla u, \nabla v \rangle, \quad u, v \in C_0^1(\mathbb{R}^d). \tag{3.2}$$

We recall (see Example 2.3.5) that (the closure of) $\mathcal{E}$ gives rise to a family T^p of sub-Markovian semigroups on L^p, $1 \leq p < \infty$. By $-\mathcal{L}_p$ we denote the generator of T^p. The operator $\mathcal{L}_p$ is called the Dirichlet operator in L^p and $\mathcal{L}_p = -\Delta - \beta \cdot \nabla$ on $C_0^\infty(\mathbb{R}^d) =: C_0^\infty$ if $\beta \in L^p_{\text{loc}}$, where $C_0^\infty(\mathbb{R}^d)$ stands for the set of smooth compactly supported functions on $\mathbb{R}^d$. Hence, an operator, defined by means of the formal differential expression $\mathcal{H}^{(b,q)}$, can be regarded as a perturbation of the Dirichlet operator by lower order terms.

The chapter is organised as follows. In section 3.1 we study the uniqueness problem for potential perturbations of the Dirichlet operators. Section 3.2 deals with the first order perturbations. The Dirichlet operators, with singular drifts and potentials are investigated in section 3.3. A priori estimates of the corresponding elliptic equations are stated and proved in section 3.4.

Throughout this chapter by B_R we denote a ball of radius R, centered at the origin, and preserve the notation of Example 2.3.5.

3.1 Potential Perturbations of Dirichlet Operators

We begin with the case $b = 0$. Let $\beta, q \in L^p_{\mathrm{loc}}$. Let $\mathcal{H}$ stand for an operator in L^p defined by

$$(\mathcal{H}u) = \mathcal{H}^{(0,q)}u = -\Delta u - \beta \cdot \nabla u + qu, \; u \in C_0^\infty,$$

By V and W we denote the real and imaginary parts of q respectively, so that $q = V + iW$.

Our aim is

(i) to construct an operator H_p such that $H_p \supset \mathcal{H}$ and $-H_p$ generates a C_0-semigroup on L^p (the operator H_p can be regarded as a perturbation of the Dirichlet operator by the complex-valued potential q);

(ii) to reveal conditions on β and q which ensure that C_0^∞ is a domain of strong uniqueness for the operator H_p.

As a particular case of the uniqueness result we find a criterion of strong uniqueness for the operator $\mathcal{L}_p$.

3.1.1 Construction of Generator and Formulation of Main Results

Let $\mathcal{N}$ be a non-negative self-adjoint operator in L^2. Let $\alpha > 0$. We say that a potential $V \in L^1_{\mathrm{loc}}$ belongs to the class $PK_\alpha(\mathcal{N})$ if the form $\langle V\varphi, \varphi\rangle$, $\varphi \in \mathcal{D}(|V|^{\frac{1}{2}})$, is $\tau_{\mathcal{N}}$-bounded and $\tau_{\mathcal{N}}$-bound equals α, where

$$\tau_{\mathcal{N}}(\varphi, \varphi) := \langle \mathcal{N}^{\frac{1}{2}}\varphi, \mathcal{N}^{\frac{1}{2}}\varphi\rangle, \quad \varphi \in \mathcal{D}(\mathcal{N}^{\frac{1}{2}})$$

(see (2.2) for the definition of form-boundedness).

We write $V = V^+ - V^-$, where $V^+ := V \vee 0$ and $V^- := -V \wedge 0$.

Let $p \geq 1$. Let $\beta, V^+, V^-, W \in L^p_{\text{loc}}$. Then the operator $\mathcal{H}$ is well-defined in L^p. Let $0 \leq \alpha < 1$. We assume that $V^- \in PK_\alpha(\mathcal{L} \dot{+} V^+)$. Therefore the operator $\mathcal{H}$ is quasi-accretive in L^p for all $p \in I := [p(\alpha), p'(\alpha)]$, where $p(\alpha) = \dfrac{2}{1+\sqrt{1-\alpha}}$, $p'(\alpha) = \dfrac{2}{1-\sqrt{1-\alpha}}$. Indeed, for all $\varphi \in C_0^\infty$ we have

$$\operatorname{Re}\langle \mathcal{H}\varphi, \varphi|\varphi|^{p-2}\rangle = \langle \nabla\varphi, \nabla(\varphi|\varphi|^{p-2})\rangle + \langle V|\varphi|^p\rangle$$
$$\geq \Big(\frac{4(p-1)}{p^2} - \alpha\Big)\|\nabla|\varphi|^{\frac{p}{2}}\|_2^2 + (1-\alpha)\langle V^+|\varphi|^p\rangle - c(\alpha)\|\varphi\|_p^p \geq -c(\alpha)\|\varphi\|_p^p.$$

A similar argument shows that the symmetric form

$$\mathcal{E}^{(V)}(u,v) := \langle \nabla u, \nabla v\rangle + \langle Vu, v\rangle, \quad u, v \in C_0^1,$$

is bounded below. Therefore the closure of $\mathcal{E}^{(V)}$ gives rise to an analytic semigroup U on L^2. The estimate

$$\|U(t) \restriction_{L^2 \cap L^p}\|_{L^p \to L^p} \leq \exp(k_p t), \; p \in I, \; t \geq 0,$$

which was established in [52, Th.3.2], implies that for every $p \in I$ we can construct a quasi-contractive C_0-semigroup U^p on L^p. Namely,

$$U^p(t) := \Big(U(t) \restriction_{L^2 \cap L^p}\Big)^{\sim}_{L^p \to L^p}, \; t \geq 0.$$

We shall write $U^p(t) = \exp(-t\mathcal{A}_p)$. One can readily see that $\mathcal{A}_p \supset \mathcal{L}_p + V \restriction_{C_0^\infty}$ since $V \in L^p_{\text{loc}}$.

It follows from Proposition 2.2.24 that the semigroup U^p is positive. The potential $W \in L^p_{\text{loc}} \subset L^1_{\text{loc}}$, and so by Proposition 2.3.6 $|W|$ is U^p-regular. Hence, Proposition 2.3.2 implies that the limit in L^p

$$S^p(t) = s\text{-}\lim_n \exp(-t(\mathcal{A}_p + iW_n))$$

exists for all $t \geq 0$ and S^p is a C_0-semigroup (recall that W_n stand for the truncation of W). By $-H_p$ we denote the generator of S^p. It was shown in subsection 2.3.2 (see the discussion after Proposition 2.3.2), that $H_p \supset \mathcal{H}$.

Next we study the uniqueness problem for the operator H_p. First we formulate an assumption on the logarithmic derivative β.

(A1) There exists an $R < \infty$ such that for every $R_1 > R$ there is a constant $C = C(R_1) \geq 0$ such that for all $\varphi \in C_0^\infty$ we have

$$\| \, |\beta| \varphi \mathbb{1}_{B_{R_1} \setminus B_R} \|_p^2 \leq C(\| \, |\nabla \varphi| \, \|_2^2 + \|\varphi\|_2^2),$$

where $\mathbb{1}_{B_{R_1} \setminus B_R}$ is the characteristic function of the set $B_{R_1} \setminus B_R$.

Unless otherwise stated further on we assume that R is the same as in **(A1)**. Let $\mathcal{F}_R$ stand for the class of spherically symmetric functions $\eta : \mathbb{R}^d \mapsto \mathbb{R}$ with properties $\eta \in C_0^\infty$, $0 \leq \eta \leq 1$, $\eta = 1$ on the ball B_R. Set $V_\eta := V\eta$, $W_\eta := W\eta$. Let $\mathcal{L}_\eta$ stand for the operator in L^2 associated with the closure of the form

$$\mathcal{E}_\eta(u, v) = \langle \eta \nabla u, \eta \nabla v \rangle, \quad u, v \in C_b^1(\mathbb{R}^d).$$

Now we formulate the conditions on the potential q.

(A2) For every $\eta \in \mathcal{F}_R$ there exists a number $a = a(\eta) > 0$ such that the potential $W_\eta \in PK_a(\mathcal{L}_\eta \dotplus V_\eta^+)$;

(A3) there exists a number $0 \leq \alpha < 1$ such that for all $\eta \in \mathcal{F}_R$ the potential $V_\eta^- \in PK_\alpha(\mathcal{L}_\eta \dotplus V_\eta^+)$;

(A4) the potential $V^- \in L_{\text{loc}}^{k(\alpha)}$, where $k(\alpha) := k_p(\alpha) := \dfrac{1 + \sqrt{1-\alpha}}{1 + \sqrt{1-\alpha} - \frac{2}{p'}}$.

One can see that $k(\alpha) > p$ if $\alpha > 0$. Therefore $V^- \in L_{\text{loc}}^p$. We note that assumption **(A3)** implies that $V^- \in PK_\alpha(\mathcal{L} \dotplus V^+)$. Indeed, for every $\varphi \in C_0^\infty$ there is a function $\eta \in \mathcal{F}_R$ such that $\eta \restriction_{\text{supp}\,\varphi} = 1$. Therefore we have

$$\langle V^- |\varphi|^2 \rangle = \langle V_\eta^- |\varphi|^2 \rangle$$

and

$$\langle \nabla \varphi, \nabla \varphi \rangle + \langle V^+ |\varphi|^2 \rangle = \langle \eta \nabla \varphi, \eta \nabla \varphi \rangle + \langle V_\eta^+ |\varphi|^2 \rangle.$$

Hence, if β, V^+, $W \in L_{\text{loc}}^p$ and conditions **(A2)**-**(A4)** are satisfied, then we can construct the generator $-H_p$ in L^p, $p \in I$ with the property $H_p \supset \mathcal{H}$.

We set $\bar{p}(\alpha) := p(\alpha) \vee \frac{3}{2}$. The first uniqueness result reads as follows.

Theorem 3.1.1. *Let $\bar{p}(\alpha) < p \leq 2$. Let $\beta \in L_{\text{loc}}^{2p}$ and V^+, $W \in L_{\text{loc}}^p$. We assume that conditions* **(A1)**-**(A4)** *hold. Then $C_0^\infty(\mathbb{R}^d)$ is a domain of strong uniqueness for the operator H_p.*

In the case when $p(\alpha) < 3/2$ we have obtained the following criterion for the strong uniqueness of H_p.

Theorem 3.1.2. *Let $p(\alpha) < p \leq 3/2$. Let $\beta \in L^{\frac{2p}{2-p}}_{\text{loc}}$ and $V^+, W \in L^p_{\text{loc}}$. We assume that conditions* **(A1)-(A4)** *hold. Then $C_0^\infty(\mathbb{R}^d)$ is a domain of strong uniqueness for the operator H_p.*

As a particular case of Theorems 3.1.1 and 3.1.2 we derive a uniqueness result for the operator $\mathcal{L}_p$.

Corollary 3.1.3. *Let $1 < p \leq 2$. Assume that condition* **(A1)** *holds and*

(i) $\beta \in L^{2p}_{\text{loc}}$, if $p > 3/2$;

(ii) $\beta \in L^{\frac{2p}{2-p}}_{\text{loc}}$, otherwise.

Then $C_0^\infty(\mathbb{R}^d)$ is a domain of strong uniqueness for the operator $\mathcal{L}_p$.

The proofs of Theorems 3.1.1 and 3.1.2 consist of two main parts. First, the problem is reduced ("localised") to the strong uniqueness of a degenerate operator with coefficients vanishing outside a ball (Theorem 3.1.4). In order to prove the strong uniqueness for the operator on the ball we employ the perturbation technique from [8] and the method of a priori estimates developed in [51] (Theorem 3.1.9). The relevant a priori estimates (Theorem 3.4.1 and Proposition 3.4.4) are given in the end of this chapter.

3.1.2 Localisation Theorem in L^p, $p < 2$

Let $H_{\eta,p}$ stand for the minus-generator of a C_0-semigroup on L^p, associated with the closure of the form

$$\mathcal{E}_\eta^{(q)}(u,v) = \langle \eta\nabla u, \eta\nabla v\rangle + \langle V_\eta u, v\rangle + i\langle W_\eta u, v\rangle, \quad u, v \in C_0^1(\mathbb{R}^d).$$

(Observe that the form $\mathcal{E}_\eta$ is sectorial due to **(A2)** and **(A3)**.) Since $\beta\eta$, V_η, $W_\eta \in L^p$ one can see that $H_{\eta,p} \supset -(\nabla + \beta)\cdot\eta^2\nabla + V_\eta + iW_\eta \restriction_{C_b^2(\mathbb{R}^d)\cap L^p}$. Set $H_\eta := H_{\eta,2}$.

The localisation result is given in Theorem 3.1.4 below.

Theorem 3.1.4. *Let $p(\alpha) < p \leq 2$. Let $\beta, V, W \in L^p_{\text{loc}}$. We assume that conditions* **(A1)-(A3)** *are satisfied. We also assume that for all $\eta \in \mathcal{F}_R$ the closure in L^p of the operator $H_{\eta,p}\restriction_{C_0^\infty}$ is m-accretive. Then C_0^∞ is a core for the operator H_p.*

Proof. The operator $H_p \upharpoonright_{C_0^\infty}$ is quasi-accretive. So, by the Lumer-Phillips theorem, it suffices to check that $\mathrm{Ran}\,(\lambda + H_p) \upharpoonright_{C_0^\infty}$ is dense in L^p for some $\lambda > 0$, i.e. we have to show that

$$u \in L^{p'} \text{ and } \langle(\lambda + H_p)\varphi, u\rangle = 0 \text{ for all } \varphi \in C_0^\infty \tag{3.3}$$

yields $u = 0$.

The proof is divided into three steps.

Step 1. Let u satisfy (3.3), η, $\xi \in \mathcal{F}_R$ and $\eta = 1$ on $\mathrm{supp}\,\xi$. Then

$$u\xi \in \mathcal{D}(\mathcal{L}_\eta^{\frac{1}{2}}) \cap \mathcal{D}((V_\eta^+)^{\frac{1}{2}}) =: \mathcal{D}.$$

Indeed, a direct computation shows that for all $\varphi \in C_0^\infty$ we have

$$\langle(\lambda + H)\varphi, u\xi\rangle = \langle(\lambda + H_\eta)\varphi, u\xi\rangle.$$

Since $\xi\varphi \in C_0^\infty$, (3.3) implies that

$$\langle(\lambda + H_\eta)\varphi, u\xi\rangle = \langle 2\nabla\xi \cdot \nabla\varphi + (\Delta\xi)\varphi + (\beta \cdot \nabla\xi)\varphi, u\rangle. \tag{3.4}$$

It follows from the Hölder inequality that

$$\begin{aligned} \|\nabla\xi \cdot \nabla\varphi\|_p &\le C_\xi \|\eta|\nabla\varphi|\,\|_2 = C_\xi \|\mathcal{L}_\eta^{\frac{1}{2}}\varphi\|_2, \\ \|(\Delta\xi)\varphi\|_p &\le C_\xi \|\varphi\|_2, \\ \|(\beta \cdot \nabla\xi)\varphi\|_p &\le C_\xi \|(1 + \mathcal{L}_\eta)^{\frac{1}{2}}\varphi\|_2 \end{aligned}$$

(we made use of assumption **(A1)** to derive the last estimate). Observing that by **(A3)**

$$\|(1 + \mathcal{L}_\eta)^{\frac{1}{2}}\varphi\|_2 \le C_\alpha \|(\lambda + \mathcal{L}_\eta \dotplus V_\eta)^{\frac{1}{2}}\varphi\|_2,$$

we conclude that

$$|\langle(\lambda + H_\eta)\varphi, u\xi\rangle| \le C_{u,\xi} \|(\lambda + \mathcal{L}_\eta \dotplus V_\eta)^{\frac{1}{2}}\varphi\|_2, \quad \forall \varphi \in C_0^\infty. \tag{3.5}$$

Since C_0^∞ is a core for the form, inequality (3.5) implies that the left-hand side of (3.4) defines a linear continuous functional on $\mathcal{D}$. Therefore by the Riesz representation theorem one can find a $v \in \mathcal{D}$ such that

$$\langle(\lambda + H_\eta)\varphi, u\xi\rangle = \langle(\lambda + \mathcal{L}_\eta \dotplus V_\eta)^{\frac{1}{2}}\varphi, (\lambda + \mathcal{L}_\eta \dotplus V_\eta)^{\frac{1}{2}}v\rangle. \tag{3.6}$$

Since the form $\mathcal{E}_\eta^{(q)}$ is sectorial and the operator $(\lambda+\mathcal{L}_\eta \dot{+} V_\eta) \geq 0$ for all $\lambda \geq c(\alpha)$ it follows from Theorem 2.1.7 that

$$\lambda + H_\eta = (\lambda+\mathcal{L}_\eta \dot{+} V_\eta)^{\frac{1}{2}}(\mathrm{Id}+i\,\mathcal{B})(\lambda+\mathcal{L}_\eta \dot{+} V_\eta)^{\frac{1}{2}},$$

where $\mathcal{B}$ is a bounded self-adjoint operator on L^2. The operator $\mathrm{Id}-i\,\mathcal{B} : L^2 \to L^2$ is clearly, a bijection and the mapping $(\lambda+\mathcal{L}_\eta \dot{+} V_\eta)^{\frac{1}{2}} : \mathcal{D} \to L^2$ is known to be an isomorphism. Therefore for every $v \in \mathcal{D}$ there exists (a unique) $w \in \mathcal{D}$ such that

$$(\mathrm{Id}-i\,\mathcal{B})(\lambda+\mathcal{L}_\eta \dot{+} V_\eta)^{\frac{1}{2}} w = (\lambda+\mathcal{L}_\eta \dot{+} V_\eta)^{\frac{1}{2}} v. \tag{3.7}$$

Combining (3.6) and (3.7) we get

$$\begin{aligned}\langle(\lambda+H_\eta)\varphi,\, u\xi\rangle &= \langle(\lambda+\mathcal{L}_\eta \dot{+} V_\eta)^{\frac{1}{2}}\varphi,\, (\mathrm{Id}-i\,\mathcal{B})(\lambda+\mathcal{L}_\eta \dot{+} V_\eta)^{\frac{1}{2}} w\rangle \\ &= \langle(\lambda+H_\eta)\varphi,\, w\rangle.\end{aligned} \tag{3.8}$$

Employing the strong uniqueness of $H_{\eta,p} \restriction_{C_0^\infty}$ and (3.8) we obtain the equality

$$\langle\psi,\, u\xi\rangle = \langle\psi,\, w\rangle,$$

for all $\psi \in L^p$. Therefore $w \in L^{p'}$ and $w = u\xi$ a.e.

Step 2. Suppose that u satisfies (3.3) and $\xi \in \mathcal{F}_R$. Then there exists a constant $C = C(\alpha, p)$ which depends only on α and p such that

$$\|u\xi\|_{p'} \leq C\|u|\nabla\xi|^{\frac{2}{p'}}\|_{p'}. \tag{3.9}$$

Let us choose $\hat{\xi}$, $\eta \in \mathcal{F}_R$ such that $\hat{\xi} \restriction_{supp\xi}= 1$ and $\eta \restriction_{supp\hat{\xi}}= 1$. Set $\hat{u} := u\hat{\xi}$ and note that $\hat{u} \restriction_{supp\xi}= u \restriction_{supp\xi}$. By Step 1 $\hat{u} \in \mathcal{D}$.

We introduce the functions $g_n(y) = y(|y| \wedge n)^{p'-2}$, $n \in \mathbb{N}$, $y \in \mathbb{C}$. For every $n \in \mathbb{N}$ the function g_n is clearly Lipschitz continuous, therefore the mappings $(p'-1)^{-1}n^{2-p'}g_n : \mathbb{C} \to \mathbb{C}$, $n \in \mathbb{N}$ are normal contractions of $\mathbb{C}$. Since $\mathcal{D}$ is a Dirichlet space it follows that $g_n \circ v \in \mathcal{D}$ for every $v \in \mathcal{D}$ (see e.g. [65], Th. XIII.51). Thus the functions $\varphi^{(n)} = g_n \circ u\xi \in \mathcal{D}$, $n \in \mathbb{N}$.

For $n \in \mathbb{N}$ let $(\varphi_k^{(n)})_{k\in\mathbb{N}} \subset C_0^\infty$ be a sequence such that $\varphi_k^{(n)} \to \varphi^{(n)}$ in $\mathcal{D}$. Due to the choice of ξ, $\hat{\xi}$ and η we have $u = \hat{u}$ and $\eta\nabla\hat{u} = \nabla\hat{u}$ on $\operatorname{supp}\xi$. We rewrite (3.4) with $\varphi = \varphi_k^{(n)}$:

$$\begin{aligned}&\langle\nabla\varphi_k^{(n)},\, \nabla(u\xi)\rangle + \langle(\lambda+V+iW)\varphi_k^{(n)},\, u\xi\rangle \\ &= \langle\nabla\varphi_k^{(n)},\, u\nabla\xi\rangle - \langle\varphi_k^{(n)},\, \nabla\hat{u}\nabla\xi\rangle.\end{aligned} \tag{3.10}$$

Passing to the limit as $k \to \infty$ in (3.10) and taking the real part of both sides of the obtained equality we get

$$\begin{aligned}&\operatorname{Re}\langle \nabla\big(u\xi(|u\xi| \wedge n)^{p'-2}\big),\, \nabla(u\xi)\rangle + \langle (\lambda + V)|u\xi|^2(|u\xi| \wedge n)^{p'-2}\rangle \\ &= \operatorname{Re}\Big(\langle \nabla\big(u\xi(|u\xi| \wedge n)^{p'-2}\big),\, u\nabla\xi\rangle - \langle u\xi(|u\xi| \wedge n)^{p'-2},\, \nabla\hat{u}(\nabla\xi)\rangle\Big). \qquad (3.11)\end{aligned}$$

We introduce the notation $u^{(n)} := (|u\xi| \wedge n)^{\frac{p'-2}{2}}$, $v^{(n)} := u^{(n)}u\xi$ (note that then $\varphi^{(n)} = (u^{(n)})^2 u\xi$).

A straightforward computation shows that

$$\begin{aligned}\nabla v^{(n)} &= u^{(n)}\Big(\nabla(u\xi) + \frac{p'-2}{2}\mathbb{1}_{-n}\operatorname{sgn} u \nabla|u\xi|\Big) \\ &= \operatorname{sgn} u\Big(u^{(n)}\operatorname{sgn} \overline{u}\nabla(u\xi) + \frac{p'-2}{2}\mathbb{1}_{-n}u^{(n)}\nabla|u\xi|\Big).\end{aligned}$$

and, analogously,

$$\nabla\varphi^{(n)} = (u^{(n)})^2 \operatorname{sgn} u(\operatorname{sgn} \overline{u}\nabla(u\xi) + (p'-2)\mathbb{1}_{-n}\nabla|u\xi|),$$

where $\mathbb{1}_n$ and $\mathbb{1}_{-n}$ stand for the characteristic functions of the sets $\{|u\xi| \geq n\}$ and $\{|u\xi| < n\}$ respectively. We set $\phi_n := u^{(n)}\operatorname{Re}(\operatorname{sgn}\overline{u}\nabla(u\xi))$ and $\psi_n := u^{(n)}\operatorname{Im}(\operatorname{sgn}\overline{u}\nabla(u\xi))$. Then

$$\operatorname{sgn}\overline{u}\nabla v^{(n)} = \Big(\frac{p'}{2}\mathbb{1}_{-n} + \mathbb{1}_n\Big)\phi_n + i\psi_n$$

and

$$\nabla\varphi^{(n)} = u^{(n)}\operatorname{sgn} u((p'-1)\mathbb{1}_{-n} + \mathbb{1}_n)(\phi_n + i\psi_n).$$

Thus

$$\operatorname{Re}\nabla\varphi^{(n)} \cdot \overline{\nabla(u\xi)} = \big((p'-1)\mathbb{1}_{-n} + \mathbb{1}_n\big)\phi_n^2 + \psi_n^2$$

and

$$|\nabla v^{(n)}|^2 = \Big(\frac{(p')^2}{4}\mathbb{1}_{-n} + \mathbb{1}_n\Big)\phi_n^2 + \psi_n^2.$$

We note that $\frac{4}{qq'}\frac{q^2}{4} = q - 1$ for all $q \geq 1$ and conclude that

$$\operatorname{Re}\langle \nabla\varphi^{(n)},\, \nabla(u\xi)\rangle = \frac{4}{pp'}\|\nabla v^{(n)}\|_2^2 + \Big(1 - \frac{4}{pp'}\Big)\langle \mathbb{1}_n\phi_n^2 + \psi_n^2\rangle. \qquad (3.12)$$

Making successive use of (3.12) and condition **(A3)** we estimate the left-hand side of (3.11) below as follows.

$$\begin{aligned}
&\operatorname{Re}\langle\nabla\big(u\xi(|u\xi|\wedge n)^{p'-2}\big),\ \nabla(u\xi)\rangle+\langle(\lambda+V)|u\xi|^2(|u\xi|\wedge n)^{p'-2}\rangle\\
&\geq\lambda\|v^{(n)}\|_2^2+\frac{4}{pp'}\|\nabla v^{(n)}\|_2^2+\langle V^+|v^{(n)}|^2\rangle-\langle V^-|v^{(n)}|^2\rangle\\
&\geq(\lambda-c(\alpha))\|v^{(n)}\|_2^2+\left(\frac{4}{pp'}-\alpha\right)\|\nabla v^{(n)}\|_2^2+(1-\alpha)\langle V^+|v^{(n)}|^2\rangle. \qquad (3.13)
\end{aligned}$$

Next we transform the right-hand side of (3.11). A straightforward computation gives

$$\begin{aligned}
&\operatorname{Re}\Big(\langle\nabla(u\xi(|u\xi|\wedge n)^{p'-2}),\ u\nabla\xi\rangle-\langle u\xi(|u\xi|\wedge n)^{p'-2},(\nabla\hat{u})\nabla\xi\rangle\Big)\\
&=2\,\frac{p'-2}{p'}\left\langle\left(\frac{\nabla\xi}{\xi}\right)|u\xi|^{\frac{p'}{2}}\nabla\big(|u\xi|\wedge n\big)^{\frac{p'}{2}}\right\rangle\\
&+\left\langle\left(\frac{\nabla\xi}{\xi}\right)^2|u\xi|^2\big(|u\xi|\wedge n\big)^{p'-2}\right\rangle. \qquad (3.14)
\end{aligned}$$

The Schwarz and Cauchy inequalities imply that the right-hand side of (3.14) is not greater than

$$\varepsilon\left\|\nabla\big(|u\xi|\wedge n\big)^{\frac{p'}{2}}\right\|_2^2+\left(\frac{1}{\varepsilon}\left(\frac{p'-2}{p'}\right)^2+1\right)\left\|\left(\frac{\nabla\xi}{\xi}\right)v^{(n)}\right\|_2^2 \qquad (3.15)$$

for any positive ε.

Combining (3.13) with (3.15) and recalling that $1-\alpha>0$ and $\|\nabla v^{(n)}\|_2\geq\|\nabla(|u\xi|\wedge n)^{\frac{p'}{2}}\|_2$, we infer that

$$\begin{aligned}
&(\lambda-c(\alpha))\|v^{(n)}\|_2^2+\left(\frac{4}{pp'}-\alpha\right)\|\nabla v^{(n)}\|_2^2\\
&\leq\varepsilon\left\|\nabla v^{(n)}\right\|_2^2+\left(\frac{1}{\varepsilon}\left(\frac{p'-2}{p'}\right)^2+1\right)\left\|\left(\frac{\nabla\xi}{\xi}\right)v^{(n)}\right\|_2^2 \qquad (3.16)
\end{aligned}$$

for all $\varepsilon>0$. Choosing $\varepsilon:=\frac{pp'}{4-\alpha pp'}>0$ we obtain the inequality

$$(\lambda-c(\alpha))\|v^{(n)}\|_2^2\leq C_p\left\|\left(\frac{\nabla\xi}{\xi}\right)v^{(n)}\right\|_2^2.$$

It follows from the B.Levi theorem that $|v^{(n)}|^2\to|u\xi|^{p'}$ as $n\to\infty$ in L^1. Therefore passing to the limit in the last estimate we complete the proof of (3.9).

Step 3. $u=0$.

We choose a sequence $(\xi_n)_{n\in\mathbb{N}}$ such that $\xi_n\to1$ pointwise and $|\nabla\xi_n|\leq1$, and see that $u\xi_n\to u$ and $u\nabla\xi_n\to0$ in $L^{p'}$. This completes the proof. □

Remark. *We suspect that condition* **(A2)** *is superfluous.*

3.1.3 Localisation Theorem in L^2

A careful analysis of the proof of Theorem 3.1.4 reveals that if $p = 2$ then in (3.9) the constant C can be chosen as $C = (\lambda - c(\alpha))^{-1}$, and so is dependent only on $c(\alpha)$. This implies that assumptions **(A3)** and **(A4)** can be relaxed, namely, we can assume that

(A3′) there exists a number $\bar{c} \in \mathbb{R}$ such that for every $\eta \in \mathcal{F}_R$ one can find a constant $0 < \alpha(\eta) < 1$ such that

$$\langle V_\eta^- \varphi, \varphi\rangle \le \alpha(\eta)\Big(\|\nabla\varphi\|_2^2 + \langle V_\eta^+ |\varphi|^2\rangle\Big) + \bar{c}\|\varphi\|_2^2;$$

(A4′) the potential $V_\eta^- \in L^{k(\eta)}$, where $k(\eta) = \dfrac{1+\sqrt{1-\alpha(\eta)}}{1+\sqrt{1-\alpha(\eta)}-\frac{2}{p'}}$.

Condition **(A3′)** implies that

$$\langle V^- \varphi, \varphi\rangle \le \|\nabla\varphi\|_2^2 + \bar{c}\|\varphi\|_2^2.$$

for all $\varphi \in C_0^\infty$. Therefore assumptions **(A2)** and **(A3′)** guarantee that the form $\mathcal{E}_\eta$ is sectorial. We also observe that $k(\eta) > 2$ and $k(\eta) \to 2$ if $\alpha(\eta) \to 0$.

Theorem 3.1.5 below is an extension of the main result of [70] to the case of weighted spaces and complex-valued potentials.

Theorem 3.1.5. *Let $p = 2$. Let β, V, $W \in L^2_{\mathrm{loc}}$ and conditions* **(A1)**, **(A2)** *and* **(A3′)** *hold. Assume that the closure in L^2 of the operator $H_\eta \restriction_{C_0^\infty}$ is m-accretive. Then the closure of $\mathcal{H}$ is m-accretive.*

Proof. It follows from **(A3′)** that the operator $\lambda + \mathcal{H}$ is accretive for all $\lambda \ge \bar{c}$. Therefore by Lumer-Phillips' theorem it suffices to verify that the range of $(\lambda+\mathcal{H})$ is dense in L^2, i.e. that if $u \in L^2$ and

$$\langle(\lambda + \mathcal{H})\varphi, u\rangle = 0 \;\text{ for all }\; \varphi \in C_0^\infty, \tag{3.17}$$

then $u = 0$.

Claim. $u\xi \in \mathcal{D} := \mathcal{D}(\mathcal{L}_\eta^{\frac{1}{2}}) \cap \mathcal{D}((V_\eta^+)^{\frac{1}{2}})$ for all ξ, $\eta \in \mathcal{F}_R$, such that $\eta = 1$ on $\operatorname{supp}\xi$. Indeed, it is easy to check that

$$\langle(\lambda + \mathcal{H})\varphi, u\xi\rangle = \langle(\lambda + H_\eta)\varphi, u\xi\rangle \;(\varphi \in C_0^\infty(\mathbb{R}^d)).$$

Since $\xi\varphi \in C_0^\infty(\mathbb{R}^d)$, (3.17) implies that

$$\langle(\lambda + \mathcal{H})\varphi,\, u\xi\rangle = 2\langle\nabla\xi \cdot \nabla\varphi,\, u\rangle + \langle(\Delta\xi + \beta \cdot \nabla\xi)\varphi,\, u\rangle. \tag{3.18}$$

Applying the Schwarz inequality and **(A1)** we have

$$\begin{aligned} \|\nabla\xi \cdot \nabla\varphi\|_2 &\le C_\xi \|\eta|\nabla\varphi|\,\|_2 \le \|(1 + \mathcal{L}_\eta)^{\frac{1}{2}})\varphi\|_2, \\ \|(\Delta\xi)\varphi\|_2 &\le C_\xi\|\varphi\|_2, \\ \|(\beta\cdot\nabla\xi)\varphi\|_2 &\le C_\xi\|(1 + \mathcal{L}_\eta)^{\frac{1}{2}})\varphi\|_2, \end{aligned}$$

Observing that by **(A3′)** one can find a constant C_η depending on η such that

$$\|(1 + \mathcal{L}_\eta)^{\frac{1}{2}}\varphi\|_2 \le C_\eta\|(\lambda + \mathcal{L}_\eta \dot{+} V_\eta)^{\frac{1}{2}}\varphi\|_2,$$

we conclude that

$$|\langle(\lambda + H_\eta)\varphi,\, u\xi\rangle| \le C_{u,\xi,\eta}\|(\lambda + \mathcal{L}_\eta \dot{+} V_\eta)^{\frac{1}{2}}\varphi\|_2. \tag{3.19}$$

Estimate (3.19) implies that the left-hand side of (3.18) defines a linear bounded functional on $\mathcal{D}$, i.e. there is a $v \in \mathcal{D}$ such that

$$\langle(\lambda + H_\eta)\varphi,\, u\xi\rangle = \langle(\lambda + \mathcal{L}_\eta \dot{+} V_\eta)^{\frac{1}{2}}\varphi,\, (\lambda + \mathcal{L}_\eta \dot{+} V_\eta)^{\frac{1}{2}}v\rangle. \tag{3.20}$$

Since the form $\mathcal{E}_\eta$ is sectorial and the operator $\lambda + \mathcal{L}_\eta \dot{+} V_\eta \ge 0$ for all $\lambda \ge \bar{c}$ it follows from Theorem 2.1.7 that

$$\lambda + H_\eta = (\lambda + \mathcal{L}_\eta \dot{+} V_\eta)^{\frac{1}{2}}(\mathrm{Id} + i\,\mathcal{B})(\lambda + \mathcal{L}_\eta \dot{+} V_\eta)^{\frac{1}{2}},$$

where $\mathcal{B}$ is a bounded self-adjoint operator in L^2. The operator $Id - i\,\mathcal{B} : L^2 \to L^2$ is clearly bijective, and the mapping $(\lambda + \mathcal{L}_\eta \dot{+} V_\eta)^{\frac{1}{2}} : \mathcal{D} \to L^2$ is known to be isomorphic. Therefore for every $v \in \mathcal{D}$ one can find a (unique) $w \in \mathcal{D}$ such that

$$(\mathrm{Id} - i\,\mathcal{B})(\lambda + \mathcal{L}_\eta \dot{+} V_\eta)^{\frac{1}{2}}w = (\lambda + \mathcal{L}_\eta \dot{+} V_\eta)^{\frac{1}{2}}v. \tag{3.21}$$

Combining (3.20) and (3.21) we get

$$\langle(\lambda + H_\eta)\varphi,\, u\xi\rangle = \langle(\mathrm{Id} + i\,\mathcal{B})(\lambda + \mathcal{L}_\eta \dot{+} V_\eta)^{\frac{1}{2}}\varphi,\, (\lambda + \mathcal{L}_\eta \dot{+} V_\eta)^{\frac{1}{2}}w\rangle.$$

We employ the strong uniqueness of $H_\eta \restriction_{C_0^\infty}$ and obtain the equality $\langle\psi,\, u\xi\rangle = \langle\psi,\, w\rangle$ for all $\psi \in L^2$. Hence, $w = u\xi$ μ-a.e. and the Claim follows.

Step 2. Let $\xi \in \mathcal{F}_R$ and u be as in (3.17). Our goal is to prove the following estimate:

$$\|u\xi\|_2^2 \leq \frac{1}{\lambda - \bar{c}} \|u|\nabla\xi|\|_2^2 \text{ for all } \xi \in \mathcal{F}_R.$$

Let us choose $\hat{\xi}$, $\eta \in \mathcal{F}_R$ such that $\hat{\xi} \restriction_{supp\xi} = 1$ and $\eta \restriction_{supp\hat{\xi}} = 1$. Set $\hat{u} := u\hat{\xi}$ and note that $\hat{u} \restriction_{supp\xi} = u \restriction_{supp\xi}$. By Step 1 $\hat{u} \in \mathcal{D}$.

Let $(\varphi_k)_{k\in\mathbb{N}} \subset C_0^\infty$ be a sequence such that $\varphi_k \to u\xi$ in $\mathcal{D}$. It follows from (3.18) that

$$\langle (\lambda + H_\eta)\varphi_k, u\xi\rangle = 2\langle \nabla\xi \cdot \nabla\varphi_k, u\rangle + \langle (\Delta\xi + \beta \cdot \nabla\xi)\varphi_k, u\rangle.$$

By Step 1

$$\langle (\lambda + H_\eta)\varphi_k, u\xi\rangle = \langle \nabla\varphi_k, \nabla(u\xi)\rangle + \langle (\lambda + V + iW)\varphi_k, u\xi\rangle.$$

Observing that $\nabla(u\varphi_k) = \varphi_k\nabla\hat{u} + u\nabla\varphi_k$ on $\operatorname{supp}\xi$ we derive the equality

$$2\langle \nabla\xi \cdot \nabla\varphi_k, u\rangle + \langle (\Delta\xi + \beta \cdot \nabla\xi)\varphi_k, u\rangle = \langle \nabla\xi \cdot \nabla\varphi_k, u\rangle - \langle \varphi_k\nabla\xi, \nabla\hat{u}\rangle.$$

Hence,

$$\langle \nabla\varphi_k, \nabla(u\xi)\rangle + \langle (\lambda + q)\varphi_k, u\xi\rangle = \langle \nabla\varphi_k, u\nabla\xi\rangle - \langle \varphi_k, \nabla\hat{u} \cdot \nabla\xi\rangle.$$

Passing to the limit as $k \to \infty$ in the last equality we get

$$\|\,|\nabla(u\xi)|\,\|_2^2 + \langle (\lambda + q)|u\xi|^2\rangle = \langle \nabla(u\xi), u\nabla\xi\rangle - \langle u\xi, \nabla\hat{u} \cdot \nabla\xi\rangle. \tag{3.22}$$

We note that $\nabla(u\xi) = u\nabla\xi + \xi\nabla\hat{u}$, take the real parts of both sides of (3.22), and obtain

$$\lambda\|u\xi\|_2^2 + \|\,|\nabla(u\xi)|\,\|_2^2 + \langle V|u\xi|^2\rangle = \|\,|u\nabla\xi|\,\|_2^2,$$

since $\langle u\xi, \nabla\hat{u} \cdot \nabla\xi\rangle = \overline{\langle (\nabla\hat{u})\,\xi, u\nabla\xi\rangle}$. Next we make use of assumption **(A3′)** and conclude that

$$\begin{aligned} &\lambda\|u\xi\|_2^2 + \|\,|\nabla(u\xi)|\,\|_2^2 + \langle V|u\xi|^2\rangle \\ &\geq (\lambda - \bar{c})\|u\xi\|_2^2 + (1 - \alpha(\eta))\Big(\|\,|\nabla(u\xi)|\,\|_2^2 + \langle V^+|u\xi|^2\rangle\Big). \end{aligned}$$

Therefore

$$(\lambda - \bar{c})\|u\xi\|_2^2 \leq \|\,|u\nabla\xi|\,\|_2^2.$$

As in the proof of Theorem 3.1.4 the last estimate yields $u = 0$. □

3.1.4 Localisation Theorem in L^p, $p > 2$

Next we study the uniqueness problem for the operator H_p when $2 < p < p'(\alpha)$. Before formulating the relevant result we have to impose additional conditions on the weight ρ.

(A5) There exist numbers $s > 1$ and $c > 0$ such that for every $w \in W^{1,2}$ the following inequality holds:

$$\|w\|_{2s} \leq c\|\nabla w\|_2,$$

where $W^{1,2}$ stands for the Sobolev space of functions from L^2 with the first order derivatives in L^2. A weight ρ, satisfying **(A5)**, is said to have the Sobolev embedding property. If $d \geq 3$ and $\rho(x) \equiv 1$, then we obtain the statement of the classical Sobolev embedding theorem with $s = \dfrac{d}{d-2}$.

It follows from [31, 1.6] that if the weight ρ belongs to the Muckenhoupt class A_2, then it satisfies **(A5)**. The Muckenhoupt class A_p, $p \geq 1$, is defined by

$$\mathrm{A}_\mathrm{p} := \left\{\rho \in L^1_{loc}(\mathbb{R}^d, dx) \mid \sup \mathrm{Vol}\,(B)^{-p}\Big(\int_B \rho dx\Big)\Big(\int_B \rho^{1/1-p} dx\Big)^{p-1} < \infty\right\},$$

where the supremum is taken over all balls in $\mathbb{R}^d$. We also note that the function $\rho(x) = |x|^\delta$ belongs to A_p iff $-n < \delta < n(p-1)$.

We set $q'(\alpha) := \dfrac{2s}{1-\sqrt{1-\alpha}}$, $q(\alpha) := 1 - \dfrac{1}{q'(\alpha)}$. Assumption **(A5)** yields the following remarkable property for the resolvent of the operator $H_{\eta,p}$.

(P1) Let $\eta \in \mathcal{F}$. For all $p \in \big((p(\alpha), p'(\alpha)\big)$ and $p \leq r < q'(\alpha)$, satisfying the condition $1/p - 1/r \leq (s-1)/s$, the operator $(\lambda + H_{\eta,p})^{-1} : L^p \to L^r$ is bounded, provided $\lambda \in \rho(H_{\eta,p})$.

before formulating the localisation theorem we modify assumption **(A1)**.

(A6) We assume that there are numbers $R > 0$ and $\gamma > 0$ such that for all $R_1 > R$ one can find a constant $C = C_{R_1} > 0$ such that

$$\|\beta\varphi\mathbb{1}B_{R_1} \setminus B_R\|^2_{q(\alpha)+\gamma} \leq C\big(\|\nabla\varphi\|^2_2 + \|\varphi\|^2_2\big) \;\text{ for all }\; \varphi \in C_0^\infty.$$

The localisation result reads as follows.

Theorem 3.1.6. *Let* $2 < p < p'(\alpha)$ *and* $\beta \in L^{2p}_{\mathrm{loc}}$, $V, W \in L^p_{\mathrm{loc}}$. *We assume that conditions* **(A2)**-**(A6)** *hold. For all* $\eta, \xi \in \mathcal{F}_R$ *(where* R *is the same as in* **(A6)***) such that* $\eta = 1$ *on* $\operatorname{supp}\xi$ *the following conditions are fulfilled.*

(i) for every $\psi \in L^1 \cap L^\infty$ *there is a sequence* $(\varphi_n,\, n \in \mathbb{N}) \subset C_0^\infty$ *such that*

(a) $s\text{-}L^p\text{-}\lim_n \varphi_n = (\lambda + H_{\eta,p})^{-1}\psi$;

(b) $w\text{-}L^p\text{-}\lim_n (\lambda + H_\eta)\varphi_n = \psi$;

(c) $\sup_n \left(\|\varphi_n\|_{2p} + \| \, |\nabla \varphi_n| \, \|_p \right) < \infty$;

(ii) the operator $\nabla\xi \cdot \nabla(H_{\eta,p}^{-1}\mathbb{1}_{B_R}\cdot) : L^\infty \to L^p$ *is bounded.*

Then C_0^∞ *is a domain of strong uniqueness for the operator* H_p.

Remark. *We note that if* C_0^∞ *is a domain of strong uniqueness for the operator* $H_{\eta,p}$ *then one can find a sequence satisfying (a) and (b) in (i). Condition (c) is additional. For this reason Theorem 3.1.6 is conditional.*

The proof of Theorem 3.1.6 is based on the following two statements.

Proposition 3.1.7. *Let* $\widehat{V}^+, \widehat{V}^-, \widehat{W} \in L^1_{\mathrm{loc}}$. *We assume that there exist constants* $\alpha \in [0,\, 1)$ *and* $a > 0$ *such that* $\widehat{V}^- \in PK_\alpha(\mathcal{L} + \widehat{V}^+)$ *and* $\widehat{W} \in PK_a(\mathcal{L} + \widehat{V}^+)$, *and* **(A5)** *holds. Set* $\mathcal{A} := \mathcal{L} + \widehat{V} + i\widehat{W}$. *Then for every* $R_0 > 0$ *there is an* $R_1 \in (R,\, \infty)$ *such that*

$$\mathbb{1}_{B_{R_0}}(\lambda + \mathcal{A})^{-1}\mathbb{1}_{B_{R_1}^c} \in L(L^p,\, L^q)$$

for all p, r, *satisfying* $q(\alpha) < p \le r < q'(\alpha)$, *and sufficiently large* λ.

Proof. Let $\zeta \in C_0^\infty$ and satisfy the following conditions: $0 \le \zeta \le 1$, $\zeta(t) = 1$, if $|t| \le 1$ and $-\zeta'(t) \le \zeta^{1-\theta}(t)$ for some $\theta \in (0,\, 1)$. We set $\sigma(x) := \zeta(\varepsilon|x|)$, $x \in \mathbb{R}^d$. We fix an arbitrary $R_0 > 0$ and choose $\varepsilon < \frac{1}{R_0}$. Then $\sigma = 1$ on B_{R_0} and $\operatorname{supp}\sigma \subset B_{R_1}$ for some $R_1 > R_0$.

Let $h \in L^2 \cap L^p$ and $\operatorname{supp} h \subset B_{R_1^c}$. We set $u := (\lambda + \mathcal{A})^{-1}h$ for some $\lambda > 0$ to be chosen later. It is easy to see that $\langle(\lambda + \mathcal{A})u,\, \sigma(\sigma u)|\sigma u|^{\nu-2}\rangle = 0$ where $\nu \ge 1$. We set $v := |\sigma u|^{\frac{\nu}{2}}$. A straightforward computation implies that

$$\begin{aligned} 0 = &\operatorname{Re}\langle(\lambda + \mathcal{A})u,\, \sigma(\sigma u)|\sigma u|^{\nu-2}\rangle = \langle(\lambda + \widehat{V}^+)|v|^2\rangle - \langle\widehat{V}^-|v|^2\rangle \\ &+ \operatorname{Re}\langle\mathcal{L}(\sigma u),\, \sigma u|\sigma u|^{\nu-2}\rangle - \frac{2(\nu-2)}{\nu}\langle\sigma^{-1}v\nabla\sigma,\, \nabla v\rangle - \langle\sigma^{-2}v^2\nabla\sigma,\, \nabla\sigma\rangle \end{aligned}$$

Taking into account the trivial estimates

$$\operatorname{Re}\left\langle\mathcal{L}(\sigma u),\, \sigma u|\sigma u|^{\nu-2}\right\rangle \ge \frac{4(\nu-1)}{\nu^2}\left\|\nabla v\right\|_2^2,$$

$$|\nabla\sigma|^2 \le \varepsilon^2\sigma^{2(1-\theta)},$$

$$\left|c\langle\sigma^{-1}v\nabla\sigma,\, \nabla v\rangle\right| \le \delta\|\nabla v\|_2^2 + \frac{c^2}{4\delta}\left\|\sigma^{-1}v\nabla\sigma\right\|_2^2$$

and making use of the form-boundedness of $\widehat{V}^-$ we arrive at the inequality

$$(\lambda - c(\alpha))\|v\|_2^2 + \left(\frac{4(\nu-1)}{\nu^2} - \alpha - \delta\right)\|\nabla v\|_2^2 + (1-\alpha)\langle V^+ v^2\rangle \leq c\langle \sigma^{-2\theta} v^2\rangle.$$

We choose $\nu \in \left(\frac{2-2\sqrt{1-\alpha}}{\alpha}, \frac{2+2\sqrt{1-\alpha}}{\alpha}\right)$ and $\delta \in \left(0, \frac{4(\nu-1)}{\nu^2} - \alpha\right)$, take $\lambda > c(\alpha)$ and see that

$$\|\nabla v\|_2^2 \leq C\langle \sigma^{-2\theta} v^2\rangle$$

Using assumption **(A5)** we estimate the left-hand side of the last inequality below and conclude that

$$\|v\|_{2s}^2 \leq C\langle \sigma^{-2\theta} v^2\rangle,$$

or

$$\|\sigma u\|_{s\nu}^\nu \leq C\langle |\sigma u|^{\nu-2\theta} u^{2\theta}\rangle \leq C\|\sigma u\|_{(\nu-2\theta)t}^{\nu-2\theta}\|u\|_{2\theta t'}^{2\theta},$$

where $\frac{1}{t} + \frac{1}{t'} = 1$. We choose t in such way that $(\nu - 2\theta)t = \nu s$ and derive the estimate

$$\|\sigma u\|_{s\nu} \leq C\|u\|_{\frac{2\theta s\nu}{s\nu-\nu+2\theta}}.$$

Finally we make use of **(P1)** and complete the proof. □

Corollary 3.1.8. *In addition to conditions of Proposition 3.1.7 we assume that the operator* $\nabla\xi \cdot \nabla(\mathcal{A}^{-1}\mathbb{1}_{B_{R_0}}) : L^\infty \to L^{p_0}$ *is bounded for some* $p_0 \in (p(\alpha), p'(\alpha))$, $R_0 > 0$ *and* $\xi \in \mathcal{F}_R$, *such that* $\xi = 1$ *on* B_{R_1} *(the number* $R_1 > R_0$ *was determined in Proposition 3.1.7). Then the operator*

$$\nabla\xi \cdot \nabla(\mathcal{A}^{-1}\mathbb{1}_{B_{R_0}}) : L^p \to L^{p_0}$$

is bounded for every $p > q(\alpha)$.

Proof. Let $(\nabla\xi \cdot \nabla)' := -\nabla\xi \cdot \nabla - (\Delta\xi + \beta \cdot \nabla\xi)$. Let $\psi \in L_c^\infty$, $\operatorname{supp}\psi \subset B_{R_0}$. Then for all $h \in C_0^\infty$ we have

$$|\langle \psi, \mathcal{A}_-^{-1}(\nabla\xi \cdot \nabla)' h\rangle| = |\langle \nabla\xi \cdot \nabla\mathcal{A}^{-1}\psi, h\rangle| \leq c\|\psi\|_\infty \|h\|_{p_0'},$$

where $-\mathcal{A}_-$ is the generator associated with $\mathcal{L} + \widehat{V} - i\widehat{W} \restriction_{C_0^\infty}$ in the same way as $-\mathcal{A}$ is with $\mathcal{L} + \widehat{V} + i\widehat{W} \restriction_{C_0^\infty}$. Therefore

$$\|\mathbb{1}_{B_{R_0}}\mathcal{A}_-^{-1}(\nabla\xi \cdot \nabla)' h\|_1 \leq c\|h\|_{p_0'}.$$

We set $u := 1\!\!1_{B_{R_0}} \mathcal{A}_-^{-1}(\nabla\xi \cdot \nabla)'h$. Since $\xi = 1$ on B_{R_1} we can repeat the proof of Proposition 3.1.7 with the function u defined as above. Hence, we arrive at the inequality

$$\|u\|_{s\nu} \le C\|u\|_{\frac{2\theta s\nu}{s\nu-\nu+2\theta}}.$$

We choose the constant θ in such way that $\dfrac{2\theta s\nu}{s\nu - \nu + 2\theta} = 1$, make use of the estimate $\|u\|_1 \le c\|h\|_{p_0'}$ and see that the operator

$$1\!\!1_{B_{R_0}} \mathcal{A}_-^{-1}(\nabla\xi \cdot \nabla)' : L^{p_0'} \to L^r$$

for all $r \in [1, q'(\alpha))$. This completes the proof. □

Now we are in a position to prove Theorem 3.1.6.

Proof of Theorem 3.1.6. The operator H_p is clearly quasi-accretive. Therefore by Theorem 2.2.8 it suffices to verify that the range of $\lambda + H_p \restriction_{C_0^\infty}$ is dense in L^p for some $\lambda > 0$, i.e. we need to show that

$$u \in L^{p'} \ \text{ and } \ \langle(\lambda + H_p)\varphi, u\rangle = 0 \ \text{ for all } \ \varphi \in C_0^\infty, \tag{3.23}$$

implies that $u = 0$. We split the rest of the proof into three steps.

Step 1. We claim that $u \in L^r_{\text{loc}}$ for every $1 \le r < q'(\alpha)$.

Let $R_0 > 0$ and R_1 is defined as in Proposition 3.1.7. Let $\xi, \eta \in \mathcal{F}_R$ such that $\xi = 1$ on B_{R_1} and $\eta = 1$ on $\operatorname{supp}\xi$. Observing that

$$\langle(\lambda + H_\eta)\varphi, u\xi\rangle = \langle(\lambda + H_p)\varphi, u\xi\rangle \ \text{ for all } \ \varphi \in C_0^\infty,$$

and using (3.23) we conclude that

$$\langle(\lambda + H_\eta)\varphi, u\xi\rangle = 2\langle\nabla\xi \cdot \nabla\varphi, u\rangle + \langle(\Delta\xi + \beta \cdot \nabla\xi)\varphi, u\rangle, \tag{3.24}$$

for all $\varphi \in C_0^\infty$.

Let $\psi \in L^1 \cap L^\infty$ and $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence satisfying condition (i) of the theorem. First we observe that by (i(a)) and (i(b)) we have

$$\langle(\lambda + H_\eta)\varphi_n, u\xi\rangle \to \langle\psi, u\xi\rangle \ \text{ and } \ \langle(\Delta\xi)\varphi_n, u\rangle \to \langle(\Delta\xi)g, u\rangle,$$

where $g := (\lambda + H_{\eta,p})^{-1}\psi$. Employing (i(c)) we conclude that there exists a subsequence $(\varphi_{n_k})_{k\in\mathbb{N}} \subset (\varphi_n)_{n\in\mathbb{N}}$ such that

$$\langle\nabla\xi \cdot \nabla\varphi_{n_k}, u\rangle \to \langle\nabla\xi \cdot \nabla g, u\rangle, \ \langle\beta \cdot \nabla\xi\varphi_{n_k}, u\rangle \to \langle\beta \cdot \nabla\xi g, u\rangle.$$

Now we rewrite (3.24) with $\varphi := \varphi_{n_k}$, pass to the limit as $k \to \infty$ and obtain the equality

$$\langle \psi,\, u\xi \rangle = 2\langle \nabla\xi \cdot \nabla g,\, u\rangle + \langle (\Delta\xi + \beta \cdot \nabla\xi) g,\, u\rangle. \tag{3.25}$$

We choose ψ with $\operatorname{supp}\psi \subset B_{R_0}$. Then one can rewrite (3.25) as follows.

$$\begin{aligned} \langle \psi,\, u\mathbb{1}_{B_{R_0}} \rangle &= 2\langle \nabla\xi \cdot \nabla(\lambda + H_{\eta,p})^{-1}\mathbb{1}_{B_{R_0}}\psi,\, u\rangle \\ &\quad + \langle \psi,\, \mathbb{1}_{B_{R_0}}(\lambda + H_{\eta,p})^{-1}\mathbb{1}_{B_{R_1}^c} u(\Delta\xi + \beta \cdot \nabla\xi)\rangle \end{aligned}$$

Making use of Proposition 3.1.7, Corollary 3.1.8 and the Hölder inequality we immediately get the estimate

$$|\langle \psi,\, u\mathbb{1}_{B_{R_0}} \rangle| \leq C(R_0,\, q)\|\psi\|_q$$

for every $q > q(\alpha)$. Therefore $u \in L^r_{\mathrm{loc}}$ for every $r < q'(\alpha)$.

Step 2. Similarly to the proof Theorem 3.1.4, our next goal is to establish that $u\xi \in \mathcal{D}$. We apply the result of Step 1 and the Hölder inequality to (3.24) and infer that the following estimates hold with some constant $C_{u,\xi} > 0$ which depends only on ξ and u:

$$\begin{aligned} |\langle \nabla\xi \cdot \nabla\varphi,\, u\rangle| &\leq \|u\nabla\xi\|_2\|\nabla\varphi\|_2 \leq C_{u,\xi}\|\eta\nabla\varphi\|_2, \\ |\langle \varphi\Delta\xi,\, u\rangle| &\leq \|u\Delta\xi\|_2\|\varphi\|_2 \leq C_{u,\xi}\|\varphi\|_2, \\ |\langle \beta \cdot \nabla\xi\varphi,\, u\rangle &\leq \|u\|_{p'}\|(\beta \cdot \nabla\xi)\varphi\|_p \leq C_{u,\xi}\|\varphi\|_2 \end{aligned} \tag{3.26}$$

(the last inequality in (3.26) follows from **(A6)**). Assumption **(A2)** implies that

$$\|(1 + \mathcal{L}_\eta)^{\frac{1}{2}}\varphi\|_2 \leq C_\alpha\|(\lambda + \mathcal{L}_\eta \dotplus V_\eta)\varphi\|_2$$

for all $\varphi \in \mathcal{D}$ with $C_\alpha > 0$. Hence, for every $\varphi \in C_0^\infty$ we have

$$|\langle (\lambda + H_\eta)\varphi,\, u\xi\rangle| \leq C_{\xi,u}\|(\lambda + \mathcal{L}_\eta \dotplus V_\eta)^{\frac{1}{2}}\varphi\|_2.$$

The rest of Step 2 is identical to the corresponding part of Step 1 of the proof of Theorem 3.1.4.

Step 3. We assume that $u \neq 0$ a.e. Let $\xi \in \mathcal{F}_R$ satisfy the following condition: $\xi(x) = \zeta\Big(\frac{|x|}{M}\Big)$, $x \in \mathbb{R}^d$, where $M \geq R \vee 1$, $\zeta \in C_0^\infty(\mathbb{R}_+)$ and $\zeta' \leq \zeta^{2-p'}$ (since $1 < p' < 2$ such a choice is possible), $0 \leq \zeta \leq 1$, $\zeta = 1$ on $[0,1]$ and $\zeta = 0$ on $[2,\infty)$.

For every $\varepsilon > 0$ the functions $h_\varepsilon(y) := y(|y| \vee \varepsilon)^{p'-2}$, $y \in \mathbb{C}$ are normal contractions of $\mathbb{C}$. Therefore $u\xi \circ h_\varepsilon =: \varphi_\varepsilon \in \mathcal{D}$ (since $\mathcal{D}$ is a Dirichlet space). As in the proof of Theorem 3.1.4 equality (3.24) holds with φ replaced by φ_ε. Let $\hat{\xi} \in \mathcal{F}_R$ be such that $\eta \restriction_{supp \hat{\xi}} = 1$ and $\hat{\xi} \restriction_{supp \xi} = 1$. Then $\hat{u} := u\hat{\xi} \in \mathcal{D}$ and $\hat{u} = u$ on $\operatorname{supp} \xi$.

Similar to Step 2 of the proof of Theorem 3.1.4 we set $u_\varepsilon := (|u\xi| \vee \varepsilon)^{\frac{p'-2}{2}}$ and $v_\varepsilon := u_\varepsilon u\xi$. Clearly, $\varphi_\varepsilon = u_\varepsilon^2 u\xi$. We compute

$$\begin{aligned}\nabla v_\varepsilon &= \operatorname{sgn} u\Big(\operatorname{sgn} \bar{u} u_\varepsilon \nabla(u\xi) + \frac{p'-2}{2}\mathbb{1}_\varepsilon u_\varepsilon \nabla|u\xi|\Big),\\ \nabla\varphi_\varepsilon &= \operatorname{sgn} u u_\varepsilon\Big(\operatorname{sgn} \bar{u} u_\varepsilon \nabla(u\xi) + (p'-2)\mathbb{1}_\varepsilon u_\varepsilon \nabla|u\xi|\Big),\end{aligned}$$

where $\mathbb{1}_\varepsilon$ and $\mathbb{1}_{-\varepsilon}$ are the characteristic functions of the sets $\{|u\xi| > \varepsilon\}$ and $\{|u\xi| \le \varepsilon\}$ respectively. Setting $\phi_\varepsilon := u_\varepsilon \mathrm{Re}\,(\operatorname{sgn} \bar{u}\nabla(u\xi)) = u_\varepsilon \nabla|u\xi|$ and $\psi_\varepsilon := u_\varepsilon \mathrm{Im}\,(\operatorname{sgn} \bar{u}\nabla(u\xi))$ we see that

$$\begin{aligned}\mathrm{Re}\,\nabla\varphi_\varepsilon \cdot \overline{\nabla(u\xi)} &= ((p'-1)\mathbb{1}_\varepsilon + \mathbb{1}_{-\varepsilon})\phi_\varepsilon^2 + \psi_\varepsilon^2,\\ |\nabla v_\varepsilon|^2 &= \Big(\frac{(p')^2}{4}\mathbb{1}_\varepsilon + \mathbb{1}_{-\varepsilon}\Big)\phi_\varepsilon^2 + \psi_\varepsilon^2\end{aligned}$$

(for details see Step 2 of the proof of Theorem 3.1.4). Hence,

$$\mathrm{Re}\,\langle \nabla\varphi_\varepsilon,\, \nabla(u\xi)\rangle = \frac{4}{pp'}\|\nabla v_\varepsilon\|_2^2 + \Big(1 - \frac{4}{pp'}\Big)\langle \mathbb{1}_{-\varepsilon}\phi_\varepsilon^2 + \psi_\varepsilon^2\rangle. \tag{3.27}$$

Making use of (3.27) and assumption **(A3)** we conclude that

$$\begin{aligned}&\mathrm{Re}\,\langle \nabla(u\xi(|u\xi| \vee \varepsilon)^{p'-2}),\, \nabla(u\xi)\rangle + \langle(\lambda + V)|u\xi|^2(|u\xi| \vee \varepsilon)^{p'-2}\rangle\\ &\ge (\lambda - c(\alpha))\|v_\varepsilon\|_2^2 + \Big(\frac{4}{pp'} - \alpha\Big)\|\nabla v_\varepsilon\|_2^2 + (1-\alpha)\langle V^+|v_\varepsilon|^2\rangle.\end{aligned} \tag{3.28}$$

Next we transform the right-hand side of (3.24) with φ replaced by φ_ε:

$$\begin{aligned}&\mathrm{Re}\,\Big(\langle \nabla\varphi_\varepsilon,\, u\nabla\xi\rangle - \langle \varphi_\varepsilon,\, \nabla\hat{u}\cdot\nabla\xi\rangle\Big)\\ &= \frac{2(p'-2)}{p'}\langle |u|^{\frac{p'}{2}}\xi^{\frac{p'}{2}-1}\nabla\xi,\, \nabla(|u\xi| \vee \varepsilon)^{\frac{p'}{2}}\rangle + \langle |u|^{p'}\xi^{p'-2}|\nabla\xi|^2\mathbb{1}_\varepsilon\rangle\\ &+ \varepsilon^{p'-2}\|u\nabla\xi\mathbb{1}_{-\varepsilon}\|_2^2.\end{aligned} \tag{3.29}$$

The Schwarz and Cauchy inequalities imply that the right-hand side of (3.29) is estimated above by

$$\delta\|\nabla(|u\xi| \vee \varepsilon)^{\frac{p'}{2}}\|_2^2 + \frac{C_p}{\delta}\langle |u|^{p'}\xi^{p'-2}|\nabla\xi|^2\mathbb{1}_\varepsilon\rangle + \varepsilon^{p'-2}\|u\nabla\xi\mathbb{1}_{-\varepsilon}\|_2^2.$$

Now employing (3.28) and choosing $\delta := 4/pp' - \alpha$ we infer that

$$\begin{aligned}&\Big(\lambda - c(\alpha)\Big)\Big\langle |u\xi|^2\big(|u\xi| \vee \varepsilon\big)^{p'-2}\Big\rangle \\ &\leq C_{p,\alpha}\big\langle |u|^{p'}\xi^{p'-2}|\nabla\xi|^2 1\!\!1_\varepsilon\big\rangle + \varepsilon^{p'-2}\big\|u\nabla\xi 1\!\!1_{-\varepsilon}\big\|_2^2. \end{aligned} \tag{3.30}$$

Next we note that $\Big\langle |u\xi|^2\big(|u\xi| \vee \varepsilon\big)^{p'-2}\Big\rangle \to \|u\xi\|_{p'}^{p'}$ as $\varepsilon \to 0$, by the dominated convergence theorem. It follows from the definition of ξ that $|\nabla\xi|^2\xi^{p'-2} \leq |\nabla\xi|$. Therefore

$$\lim_{\varepsilon\to 0}\big\langle |u|^{p'}\xi^{p'-2}|\nabla\xi|^2 1\!\!1_\varepsilon\big\rangle \leq \langle |u|^{p'}|\nabla\xi|\rangle.$$

Finally,

$$\varepsilon^{p'-2}\big\|u\nabla\xi 1\!\!1_{-\varepsilon}\big\|_2^2 \leq \varepsilon^{p'}\|\nabla\xi\|_2^2 \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Hence, passing to the limit as $\varepsilon \to 0$ in (3.30 and taking $\lambda = c(\alpha) + 1$ we obtain the estimate

$$\|u\xi\|_{p'} \leq C_{\alpha,p}\|u|\nabla\xi|^{\frac{1}{p'}}\|_{p'}.$$

Repeating the argument in Step 3 of the proof of Theorem 3.1.4 we show that $u = 0$. □

3.1.5 Uniqueness for Degenerate Operators

Now we are heading towards formulating and proving the uniqueness result for the operator $H_{\eta,p}$.

Before proceeding further we are discussing some properties of degenerate operators with smooth coefficients. Let Ω stand for the interior of $\operatorname{supp}\eta$ and $b \in C^\infty(\overline{\Omega})$. Let us consider the operator $\mathcal{A}_\eta = -(\nabla + b)\cdot\eta^2\nabla$ in $C(\overline{\Omega})$ with the domain $\mathcal{D}(\mathcal{A}_\eta) = C^2(\overline{\Omega})$. By a result of Taira ([73, Th.1]) the closure $\overline{\mathcal{A}_\eta}$ of $\mathcal{A}_\eta \restriction_{C^2(\overline{\Omega})}$ generates a Feller semigroup (i.e. a strongly continuous semigroup of contractions) $e^{-t\overline{\mathcal{A}_\eta}}$ on $C(\overline{\Omega})$.

Theorem 3.1.9. *Let $\bar{p}(\alpha) < p \leq 2$. Let $\beta \in L^{2p}_{\mathrm{loc}}$ and $V^+, W \in L^p_{\mathrm{loc}}$. Assume that conditions* **(A2)-(A4)** *hold. Then C_0^∞ is a core for the operator $H_{\eta,p}$ in L^p.*

Remark. *If $p = 2$ assumptions* **(A3)** *and* **(A4)** *are to be replaced by* **(A3′)** *and* **(A4′)**, *respectively.*

Proof. By $\widehat{\mathcal{L}}_{\eta,p}$, $\widehat{H}_{\eta,p}$ and $\mathcal{N}^{\pm}_{\eta,p}$ we denote the generators of C_0-semigroups in $L^p(\Omega) := L^p(\Omega, \rho dx)$ associated with the forms $\langle\eta\nabla u, \eta\nabla v\rangle$, $\langle\eta\nabla u, \eta\nabla v\rangle + \langle q_\eta u, v\rangle$

and $\langle \eta\nabla u, \eta\nabla v\rangle + \langle (V_\eta^+ + W_\eta)u, v\rangle$, $u, v \in C_0^1(\Omega)$, respectively. It follows from Lemma 2.3.7 that

$$\exp(-t\mathcal{N}_{\eta,p}^{\pm}) = s\text{-}\lim_{n,m} \exp(-t(\widehat{\mathcal{L}}_{\eta,p} + V_{\eta n} \pm iW_{\eta m}),$$

where $V_{\eta n}$, $n \in \mathbb{N}$, and $W_{\eta m}$, $m \in \mathbb{N}$, are the truncations of V_η and W_η respectively. Therefore Lemma 2.3.4 implies that

$$(\lambda + \mathcal{N}_\eta^{\pm})^{-1} L^\infty(\Omega) \subset \mathcal{D}(\widehat{\mathcal{L}}_{\eta,p}) \cap \mathcal{D}(q_\eta).$$

We divide the proof into several steps.

Step 1. First we prove that the set $\mathcal{D}_1 := (\lambda + \mathcal{N}_{\eta,p}^+)^{-1} C_0^\infty(\Omega)$ is a core for $\widehat{H}_{\eta,p}$. As in the proof of Theorem 3.1.5 it suffices to check that if $u \in L^{p'}(\Omega)$ and $\langle (\lambda + \widehat{H}_{\eta,p})\varphi, u\rangle = 0$ for all $\varphi \in \mathcal{D}_1$ then $u = 0$. (Observe that here and below $\langle g\rangle := \int_\Omega g(x)\rho dx$.) Hence, we obtain the equality

$$\langle (\lambda + \mathcal{N}_{\eta,p}^+)\varphi, u\rangle = \langle \varphi, V_\eta^- u\rangle, \quad \text{for all } \varphi \in \mathcal{D}_1. \tag{3.31}$$

Taking $\varphi = (\lambda + \mathcal{N}_{\eta,p}^+)^{-1}\psi$, $\psi \in C_0^\infty(\Omega)$ we rewrite (3.31) as follows

$$\langle \psi, u\rangle = \langle (\lambda + \mathcal{N}_{\eta,p}^+)^{-1}\psi, V_\eta^- u\rangle \ \text{ for all } \ \psi \in C_0^\infty(\Omega). \tag{3.32}$$

We observe that $V_\eta^- u \in L^q(\Omega)$ with $q = \frac{p'k(\alpha)}{p'+k(\alpha)}$. It is easy to see that for a function $g \in L^p(\Omega) \cap L^s(\Omega)$ one has $(\lambda + \mathcal{N}_{\eta,p}^-)^{-1} g = (\lambda + \mathcal{N}_{\eta,s}^-)^{-1} g$. Therefore, (3.32) yields

$$\langle \psi, u\rangle = \langle (\lambda + \mathcal{N}_{\eta,q'}^+)^{-1}\psi, V_\eta^- u\rangle,$$

where $q' := q(q-1)^{-1}$, or

$$\langle \psi, u\rangle = \langle \psi, (\lambda + \mathcal{N}_{\eta,q}^-)^{-1} V_\eta^- u\rangle \ \text{ for all } \ \psi \in C_0^\infty(\Omega).$$

Hence, we get

$$\langle \varphi, (\lambda + \mathcal{N}_{\eta,q}^-)\, u\rangle = \langle \varphi, V_\eta^- u\rangle,$$

for every $\varphi \in L^{q'}(\Omega)$. We take $\varphi = u|u|^{z-2}$ with $z = 1 + \frac{p'}{q'}$ in the last equality. Then $\varphi \in L^{q'}(\Omega)$ and we obtain

$$\langle u|u|^{z-2}, (\lambda + \mathcal{N}_{\eta,q}^-)\, u\rangle = \langle u|u|^{z-2}, V_\eta^- u\rangle. \tag{3.33}$$

Let $(u_k)_{k\in\mathbb{N}} \subset C_0^\infty(\Omega)$ be such that $u_k \to u$ strongly in $L^{p'}(\Omega)$ as $k \to \infty$. We denote by $T_q(t)$, $t \geq 0$, the C_0-semigroup generated by $\mathcal{N}_{\eta,q}^-$ and for $n \in \mathbb{N}$ set

$u_{n,k} := T_q(1/n)u_k$, $k \in \mathbb{N}$. Then $u_{n,k} \in \mathcal{D}(\mathcal{N}_\eta^-) \cap L^\infty(\Omega)$ and $\mathcal{N}_{\eta,2}^- u_{n,k} = \mathcal{N}_{\eta,q}^- u_{n,k}$. Set $\varphi_{n.k} := u_{n,k}|u_{n,k}|^{z-2}$.

One can check directly that

$$s\text{-}L^r\text{-}\lim_n \lim_k u_{n,k} = u \quad \text{for every} \quad r \in [1, p'],$$
$$s\text{-}L^q\text{-}\lim_n \lim_k \mathcal{N}_{\eta,q}^- u_{n,k} = \mathcal{N}_{\eta,q}^- u, \quad \text{and} \quad s\text{-}L^{q'}\text{-}\lim_n \lim_k \varphi_{n.k} = \varphi.$$

Let $v := u|u|^{\frac{z-2}{2}}$, $v_{n,k} := u_{n,k}|u_{n,k}|^{\frac{z-2}{2}}$. Then the above implies that

$$s\text{-}L^2\text{-}\lim_n \lim_k v_{n,k} = v \text{ and } s\text{-}L^2\text{-}\lim_n \lim_k (V_\eta^-)^{\frac{1}{2}} v_{n,k} = (V_\eta^-)^{\frac{1}{2}} v.$$

Therefore the following equalities hold (as usual $\mathcal{N}_\eta^- := \mathcal{N}_{\eta,2}^-$ and $\widehat{\mathcal{L}}_\eta := \widehat{\mathcal{L}}_{\eta,2}$).

$$\lim_n \lim_k \langle \varphi_{n,k}, (\lambda + \mathcal{N}_\eta^-)\, u_{n,k} \rangle = \langle \varphi, (\lambda + \mathcal{N}_{\eta,q}^-)\, u \rangle$$

and

$$\lim_n \lim_k \langle \varphi_{n,k}, V_\eta^- u_{n,k} \rangle = \langle \varphi, V_\eta^- u \rangle. \tag{3.34}$$

Let $\widehat{\mathcal{D}} := \mathcal{D}(\widehat{\mathcal{L}}_\eta^{\frac{1}{2}})) \cap \mathcal{D}((V_\eta^+)^{\frac{1}{2}})$, $n, k \in \mathbb{N}$. We introduce the functions $\varphi_{n,k,\varepsilon} := u_{n,k}(|u_{n,k}| \vee \varepsilon)^{z-2}$, $v_{n,k,\varepsilon} := u_{n,k}(|u_{n,k}| \vee \varepsilon)^{\frac{z-2}{2}}$, $\varepsilon > 0$. It follows from the dominated convergence theorem that

$$s\text{-}L^{q'}\text{-}\lim_{\varepsilon \to 0} \varphi_{n,k,\varepsilon} = \varphi_{n.k} \text{ and } s\text{-}L^2\text{-}\lim_{\varepsilon \to 0} v_{n,k,\varepsilon} = v_{n,k}.$$

Therefore

$$\lim_{\varepsilon \to 0} \langle \varphi_{n,k,\varepsilon}\, , (\lambda + \mathcal{N}_\eta^-)\, u_{n,k} \rangle = \langle \varphi_{n,k}\, , (\lambda + \mathcal{N}_\eta^-)\, u_{n,k} \rangle.$$
$$\lim_{\varepsilon \to 0} \langle V_\eta^- |v_{n,k,\varepsilon}|^2 \rangle = \langle V_\eta^- |v_{n,k}|^2 \rangle. \tag{3.35}$$

For every $\varepsilon > 0$ the function $h(y) = y(|y| \vee \varepsilon)^{z-2}$, $y \in \mathbb{C}$, is Lipschitz continuous. Therefore both $\varphi_{n,k,\varepsilon}$ and $v_{n,k,\varepsilon}$ belong to $\widehat{\mathcal{D}}$ (see Step 2 of Theorem 3.1.4). Repeating the arguments from Step 3 of the proof of Theorem 3.1.6 we get

$$\begin{aligned} \operatorname{Re} \langle \varphi_{n,k,\varepsilon}, \mathcal{N}_\eta^-\, u_{n,k} \rangle &= \langle \eta \nabla \varphi_{n,k,\varepsilon}, \eta \nabla u_{n,k} \rangle + \langle \varphi_{n,k,\varepsilon}, V_\eta^+ u_{n,k} \rangle \\ &\geq \frac{4(z-1)}{z^2} \| \eta |\nabla v_{n,k,\varepsilon}| \|_2^2 + \langle V_\eta^+ |v_{n,k,\varepsilon}|^2 \rangle. \end{aligned} \tag{3.36}$$

Combining (3.35) and (3.36) we obtain the inequality

$$\lim_n \lim_k \lim_\varepsilon \lambda \|v_{n,k,\varepsilon}\|_2^2 + \frac{4}{zz'} \|\eta |\nabla v_{n,k,\varepsilon}| \|_2^2 + \langle V_\eta^+ |v_{n,k,\varepsilon}|^2 \rangle - \langle V_\eta^- |v_{n,k,\varepsilon}|^2 \rangle \leq 0.$$

We use **(A3)** and the identity $\frac{4(z-1)}{z^2} = \alpha$ to conclude that

$$\lim_n \lim_k \lim_\varepsilon (\lambda - c(\alpha))\|v_{n,k,\varepsilon}\|_2^2 = (\lambda - c(\alpha))\|u\|_z^z \leq 0.$$

Choosing $\lambda > c(\alpha)$ we see that $u = 0$. Hence by the Lumer-Philips theorem the set $(\lambda + \mathcal{N}_{\eta,p}^+)^{-1}C_0^\infty(\Omega)$ is a core for the operator $\widehat{H}_{\eta,p}$.

Step 2. Next we show that the set $\bigcup_{m\in\mathbb{N}}(m+\widehat{\mathcal{L}}_{\eta,p})^{-1}C(\overline{\Omega})$ is a domain of strong uniqueness for $\widehat{H}_{\eta,p}$.

Indeed, by Step 1 the set $(\lambda + \mathcal{N}_{\eta,p}^+)^{-1}L^\infty(\Omega) =: \mathcal{D}_2$ is a core for the operator $\widehat{H}_{\eta,p}$ for all $\lambda > c(\alpha)$. It follows from Lemma 2.3.4 that for every $\varphi \in \mathcal{D}_2$ we have $\widehat{H}_{\eta,p}\varphi = \widehat{\mathcal{L}}_{\eta,p}\varphi + q_\eta\varphi$. Let $f \in L^\infty(\Omega)$. Then the following equalities hold.

$$\begin{aligned}
&s\text{-}L^p\text{-}\lim_m m(m+\widehat{\mathcal{L}}_{\eta,p})^{-1}(\lambda+\mathcal{N}_{\eta,p}^+)^{-1}f = (\lambda+\mathcal{N}_{\eta,p}^+)^{-1}f,\\
&s\text{-}L^p\text{-}\lim_m m\widehat{\mathcal{L}}_{\eta,p}(m+\widehat{\mathcal{L}}_{\eta,p})^{-1}(\lambda+\mathcal{N}_{\eta,p}^+)^{-1}f = \mathcal{L}_{\eta,p}(\lambda+\mathcal{N}_{\eta,p}^+)^{-1}f,\\
&s\text{-}L^p\text{-}\lim_m m(V_\eta+iW_\eta)(m+\widehat{\mathcal{L}}_{\eta,p})^{-1}(\lambda+\mathcal{N}_{\eta,p}^+)^{-1}f\\
&= (V_\eta+iW_\eta)(\lambda+\mathcal{N}_{\eta,p}^+)^{-1}f,
\end{aligned}$$

where the first equality follows from the second resolvent identity. Thus the set $\bigcup_{m\in\mathbb{N}}(m+\widehat{\mathcal{L}}_{\eta,p})^{-1}(\lambda+\mathcal{N}_{\eta,p}^+)^{-1}L^\infty(\Omega)$ is a core for the operator $\widehat{H}_{\eta,p}$. Therefore so is the set $\bigcup_{m\in\mathbb{N}}(m+\widehat{\mathcal{L}}_{\eta,p})^{-1}L^\infty(\Omega)$.

For any $f \in L^\infty(\Omega)$ one can find a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset C(\overline{\Omega})$ such that $\sup_k \|\varphi_k\|_\infty < \infty$ and $\|\varphi_k - f\|_p \to 0$ as $k \to \infty$. The operators $\widehat{\mathcal{L}}_{\eta,p}(m+\widehat{\mathcal{L}}_{\eta,p})^{-1}$, $m \in \mathbb{N}$ are clearly bounded in $L^p(\Omega)$ for every $1 \leq p < \infty$. Therefore $\bigcup_{m\in\mathbb{N}}(m+\widehat{\mathcal{L}}_{\eta,p})^{-1}C(\overline{\Omega})$ is a core for the operator $\widehat{H}_{\eta,p}$. This completes the proof of Step 2.

Step 3. $C^2(\overline{\Omega})$ is a domain of strong uniqueness for $\widehat{H}_{\eta,p}$ in $L^p(\Omega)$.

In order to prove that the set $C^2(\overline{\Omega})$ is a core for the operator $\widehat{H_{\eta,p}}$ we need to approximate an element of the form $(\lambda+\widehat{\mathcal{L}}_{\eta,p})^{-1}f$, $f \in C(\overline{\Omega})$, $\lambda > 0$ with functions from $C^2(\overline{\Omega})$ in the graph norm of $\widehat{H_{\eta,p}}$. We construct this approximating sequence in the following way. Let $\lambda > 0$. Choose $(\beta^{(n)}) \subset C^\infty(\overline{\Omega})$ such that $\beta^{(n)} \to \beta 1\!\!1_\Omega$ in $L^{2p}(\Omega)$ as $n \to \infty$ (such a sequence exists due to our assumption $\beta \in L^{2p}_{\text{loc}}$). Denote by $\mathcal{A}_{\eta n}$ the operator $\mathcal{A}_\eta$ with $b = \beta^{(n)}$. Since the closure $\overline{\mathcal{A}}_{\eta n}$ of $\mathcal{A}_{\eta n} \upharpoonright_{C^2(\overline{\Omega})}$ generates a Feller semigroup, the set $\mathcal{F}_n := (\lambda + \mathcal{A}_{\eta n})C^2(\overline{\Omega})$ is dense in $C(\overline{\Omega})$ and therefore in $L^p(\Omega)$. For every $n \in \mathbb{N}$ we take a sequence $(f_k^{(n)})_{k\in\mathbb{N}} \subset \mathcal{F}_n$ such that $f_k^{(n)} \to f$ in $C(\overline{\Omega})$ as $k \to \infty$. By $\phi_k^{(n)}$ we denote the solution of the equation

$(\lambda + \mathcal{A}_{\eta n})\phi_k^{(n)} = f_k^{(n)}$, $k \in \mathbb{N}$. Then $\phi_k^{(n)} \in C^2(\overline{\Omega}) \subset \mathcal{D}(\widehat{H}_{\eta,p}) \cap \mathcal{D}(\overline{\mathcal{A}}_{\eta n})$ and the following equalities hold in $L^p(\Omega, \rho dx)$:

$$\begin{aligned}
&(\lambda + \widehat{\mathcal{L}_{\eta,p}})^{-1} f - \phi_k^{(n)} = (\lambda + \widehat{\mathcal{L}_{\eta,p}})^{-1}(f - f_k^{(n)}) + (\lambda + \widehat{\mathcal{L}_{\eta,p}})^{-1} f_k^{(n)} - \phi_k^{(n)} \\
&= (\lambda + \widehat{\mathcal{L}_{\eta,p}})^{-1}(f - f_k^{(n)}) + (\lambda + \widehat{\mathcal{L}_{\eta,p}})^{-1}(\mathcal{A}_{\eta n} - \widehat{\mathcal{L}_{\eta,p}})(\lambda + \bar{\mathcal{A}}_{\eta n})^{-1} f_k^{(n)} \\
&= (\lambda + \widehat{\mathcal{L}_{\eta,p}})^{-1}(f - f_k^{(n)}) + (\lambda + \widehat{\mathcal{L}_{\eta,p}})^{-1}(\beta - \beta^{(n)}) \cdot \eta^2 \nabla \phi_k^{(n)},
\end{aligned} \tag{3.37}$$

and

$$f - (\lambda + \widehat{\mathcal{L}_{\eta,p}})\phi_k^{(n)} = f - f_k^{(n)} + (\beta - \beta^{(n)}) \cdot \eta \nabla \phi_k^{(n)}. \tag{3.38}$$

Next we choose n_0 and k_0 in such way that $\| |\beta^{(n)}\eta| \|_{2p} \le 2 \| |\beta\eta| \|_{2p}$ for all $n \ge n_0$ and $\|f_k^{(n)}\|_\infty \le 2\|f\|_\infty$ for every $k \ge k_0$. We employ Theorem 3.4.1 and obtain the estimate

$$\|\eta \nabla \phi_k^{(n)}\|_{2p} \le C_{p,\lambda}(\|f\|_\infty + \| |\beta\eta| \|_{2p}),$$

with C depending only on p, λ and η. Passing to the limit in k and then in n in (3.37) and (3.38) we complete the proof of Step 3.

Step 4. Now we show that C_0^∞ is a domain of strong uniqueness for the operator $H_{\eta,p}$.

By the Lumer-Philips theorem strong uniqueness of $\widehat{H_{\eta,p}}$ on $C^2(\overline{\Omega})$ is equivalent to the fact that $Ran(1 + \widehat{H_{\eta,p}}) \restriction_{C^2(\overline{\Omega})}$ is dense in $L^p(\Omega)$. Therefore, Ran $(1 + H_{\eta,p}) \restriction_{C_b^2(\mathbb{R}^d) \cap L^p}$ is dense in L^p, i.e. $C_b^2(\mathbb{R}^d) \cap L^p$ is a core for the operator $H_{\eta,p}$. Indeed, for any $\varepsilon > 0$, $f \in L^p$ we can choose $\tilde{v_1} \in C^2(\overline{\Omega})$ such that $\|[(1 + \widehat{H_{\eta,p}})\tilde{v_1} - f]\mathbb{1}_\Omega\|_p < \varepsilon/2$. Let $v \in C_b^2(\mathbb{R}^d) \cap L^p$ be an extension of $\tilde{v_1}$ and $v_2 \in C_0^2(\Omega^c)$, where Ω^c stands for the complement of Ω. The equality

$$(1 + H_{\eta,p})(v_1 + v_2) - f = [(1 + \widehat{H_{\eta,p}})\mathbb{1}_\Omega v_1 - \mathbb{1}_\Omega f] + [v_2 + \mathbb{1}_{\Omega^c}(v_1 - f)]$$

shows that $\|(1 + H_{\eta,p})v - f\|_p < \varepsilon$, provided we choose the function v_2 to satisfy the estimate $\|v_2 + \mathbb{1}_{\Omega^c}(v_1 - f)\|_p < \varepsilon/2$.

In order to check that $\overline{H_\eta \restriction_{C_b^2(\mathbb{R}^d) \cap L^p}} = \overline{H_\eta \restriction_{C_b^\infty(\mathbb{R}^d) \cap L^p}}$ we use a standard approximation for functions from $C_b^2(\mathbb{R}^d)$ by elements of $C_b^\infty(\mathbb{R}^d)$.

Finally taking a sequence $(\omega_n) \subset C_0^\infty(\mathbb{R}^d)$ such that $\omega_n \to 1$ pointwise, $|\nabla \omega_n| \le 1$ and $|\Delta \omega_n| \le 1$ one can easily show that $\omega_n \phi \to \phi$ and $H_{\eta,p}\omega_n\phi \to H_{\eta,p}\phi$ as $n \to \infty$, i.e. $C_0^\infty(\mathbb{R}^d)$ is also a core for the operator $H_{\eta,p}$. □

Proof of Theorem 3.1.2. The proof goes along the same lines as that of Theorem 3.1.1. The only difference is that in Step 3 of the proof of Theorem 3.1.9 we apply Proposition 3.4.4 instead of Theorem 3.4.1.

3.2 First Order Perturbations of Dirichlet Operators

Now we turn to the strong uniqueness problem for the generator associated with the differential expression $\mathcal{H}^{(b,0)}$ (see (3.1) for the definition of $\mathcal{H}^{(b,q)}$).

3.2.1 Construction of Generator and Formulation of Main Result

We recall that $L^p := L^p(\mathbb{R}^d, \rho\, dx)$, $p \geq 1$, where $\rho > 0$ a.e. and $\rho \in L^1_{\text{loc}}(\mathbb{R}^d, dx)$, $\beta : \mathbb{R}^d \to \mathbb{R}^d$ is the logarithmic derivative of the measure $\rho\, dx$, and $b : \mathbb{R}^d \to \mathbb{R}^d$ is a measurable vector field.

We set $\mathbf{b} := b - \beta$ and assume that $\mathbf{b} \in L^p_{\text{loc}}$. Then an operator $\mathcal{H}$, given by

$$\mathcal{H}u = \mathcal{H}^{(b,0)}u, \; u \in C_0^\infty,$$

is well-defined in L^p.

Let $\alpha \in [0, 1)$. If the potential $|b|^2 \in PK_\alpha(\mathcal{L})$, then the operator $\mathcal{H}$ is quasi-accretive in L^p for all $p \geq \dfrac{2}{2 - \sqrt{\alpha}} =: r(\alpha)$.

Indeed, for every $\varphi \in C_0^\infty$ we have

$$\begin{aligned}
&\operatorname{Re} \langle \mathcal{H}\varphi, \varphi|\varphi|^{p-2} \rangle = \langle \nabla\varphi, \nabla(\varphi|\varphi|^{p-2}) \rangle + \operatorname{Re} \langle b \cdot \nabla\varphi, \varphi|\varphi|^{p-2} \rangle \\
&\geq \frac{4(p-1)}{p^2} \| \nabla |\varphi|^{\frac{p}{2}} \|_2^2 - \delta \| \nabla |\varphi|^{\frac{p}{2}} \|_2^2 - \frac{1}{\delta p^2} \langle |b|^2 |\varphi|^p \rangle \\
&\geq \Big(\frac{4(p-1)}{p^2} - \delta - \frac{\alpha}{\delta p^2} \Big) \| \nabla |\varphi|^{\frac{p}{2}} \|_2^2 - \frac{c(\alpha)}{\delta p^2} \|\varphi\|_p^p \\
&\geq \frac{4p - 2p\sqrt{\alpha} - 4}{p^2} \| \nabla |\varphi|^{\frac{p}{2}} \|_2^2 - \frac{c(\alpha)}{\sqrt{\alpha} p} \|\varphi\|_p^p \geq - \frac{c(\alpha)}{\sqrt{\alpha} p} \|\varphi\|_p^p,
\end{aligned}$$

provided $p \geq r(\alpha)$. We note that this reasoning is valid for all $\alpha \in [0, 4)$. If however, $\alpha \geq 1$, then $r(\alpha) \geq 2$, whereas our aim is to investigate the uniqueness problem for $p \leq 2$. For this reason we restrict ourselves to the case $\alpha \in [0, 1)$.

A similar argument shows that the sesquilinear form

$$\mathcal{E}^{(b)}(u, v) = \langle \nabla u, \nabla v \rangle + \langle b \cdot \nabla u, v \rangle, \quad u, v \in C_0^1,$$

is sectorial. Let H stand for the m-sectorial operator associated with the closure of $\mathcal{E}^{(b)}$ (Theorem 2.1.4). By U we denote the analytic semigroup on L^2, generated by the operator $-H$.

A careful analysis of the proof of Theorem 4.2 in [52] shows that the following result holds.

Theorem 3.2.1. *Let $|b|^2 \in PK_\alpha(\mathcal{L})$ with some $0 \le \alpha < 1$. Then for all $f \in L^2 \cap L^p$, $p \ge r(\alpha)$, the following estimate holds:*

$$\|U(t)f\|_p \le \exp(k_p t)\|f\|_p.$$

Hence, for all $p \ge r(\alpha)$ the family of operators $U^p(t) := \big(U(t) \restriction_{L^2 \cap L^p}\big)^{\sim}_{L^p \to L^p}$, $t \ge 0$, is a quasi-contractive C_0-semigroup on L^p.

By $-H_p$ we denote the generator of the semigroup U^p constructed in Theorem 3.2.1. One can readily see that $H_p \supset \mathcal{H}$.

We impose the following conditions on the coefficients of the operator $\mathcal{H}$.

(B1) There exists an $R < \infty$ such that for every $R_1 > R$ there is a constant $C = C(R_1) \ge 0$ such that for all $\varphi \in C_0^\infty$ we have

$$\Big\| \, |\mathbf{b}|\varphi \mathbb{1}_{B_{R_1} \setminus B_R} \Big\|_p^2 \le C\big(\| \, |\nabla\varphi| \, \|_2^2 + \|\varphi\|_2^2\big),$$

where $\mathbb{1}_{B_{R_1} \setminus B_R}$ is the characteristic function of the set $B_{R_1} \setminus B_R$;

(B2) for all $\eta \in \mathcal{F}_R$ the potential $|\eta b|^2 \in PK_\alpha(\mathcal{L}_\eta)$ with some $0 \le \alpha < 1$, where $\mathcal{F}_R$ is defined in the same way as in the previous section, with R being determined by **(B1)**.

Condition **(B1)** is similar to **(A1)** in the previous section. We also note that assumption **(B2)** yields $|b|^2 \in PK_\alpha(\mathcal{L})$. Indeed, for every $\varphi \in C_0^\infty$ there is a function $\eta \in \mathcal{F}_R$ such that $\eta \restriction_{\operatorname{supp}\varphi} = 1$. Therefore we have

$$\langle b \cdot \eta^2 \nabla\varphi, \, \varphi|\varphi|^{p-2} \rangle = \langle b \cdot \nabla\varphi, \, \varphi|\varphi|^{p-2} \rangle.$$

Thus **(B2)** implies that one can construct the generator $-H_p$ of a C_0-semigroup on L^p for all $p \ge r(\alpha)$.

Next we formulate the uniqueness results for the operator H_p.

Theorem 3.2.2. *Let β, $b \in L^4_{\mathrm{loc}}$. We assume that conditions* **(B1)**-**(B2)** *hold. Then $C_0^\infty(\mathbb{R}^d)$ is a domain of strong uniqueness for the operator H.*

Set $\bar{r}(\alpha) := r(\alpha) \vee \frac{3}{2}$.

Theorem 3.2.3. *Let $\bar{r}(\alpha) < p < 2$. Let β, $b \in L^{2p}_{\mathrm{loc}}$. We assume that conditions* **(B1)-(B2)** *hold and $c(\alpha) = 0$ in* **(B2)**. *Then $C_0^\infty(\mathbb{R}^d)$ is a domain of strong uniqueness for the operator H_p.*

In the case $r(\alpha) < 3/2$ the uniqueness result for the operator H_p reads as follows.

Theorem 3.2.4. *Let $r(\alpha) < p \le 3/2$. Let $\beta \in L^2_{\mathrm{loc}}$, $\mathrm{b} \in L^{\frac{2p}{2-p}}_{\mathrm{loc}}$. We assume that conditions* **(B1)-(B2)** *hold and $c(\alpha) = 0$ in* **(B2)**. *Then $C_0^\infty(\mathbb{R}^d)$ is a domain of strong uniqueness for the operator H_p.*

In order to establish the strong uniqueness for the operator H_p we employ the same technique as in the previous section. Namely, we first "localise" the problem to that for a degenerate operator on a ball (Theorem 3.2.5), and then investigate the degenerate operator using the a priori estimates obtained in Theorem 3.4.1 and Proposition 3.4.4.

3.2.2 Proof of Uniqueness

We begin with the "localisation" theorems. For $\eta \in \mathcal{F}_R$ let $H_{\eta,p}$ stand for the minus-generator, associated with the closure of the form

$$\mathcal{E}_\eta^{(b)}(u, v) = \langle \eta\nabla u, \eta\nabla v\rangle + \langle b \cdot \eta^2 \nabla u, v\rangle, \quad u, v \in C_0^1.$$

As usual $H_\eta := H_{\eta,2}$. One can readily see that $H_{\eta,p} \supset -\nabla \cdot \eta^2 \nabla + \mathrm{b} \cdot \eta^2 \nabla \restriction_{C_b^2 \cap L^p}$.

First we prove the following conditional result.

Theorem 3.2.5. *Let $r(\alpha) < p < 2$. Assume that β, $b \in L^p_{\mathrm{loc}}$, conditions* **(B1)-(B2)** *hold and $c(\alpha) = 0$ in* **(B2)**. *We also assume that the closure in L^p of the operator $H_{\eta,p} \restriction_{C_0^\infty}$ is m-accretive. Then C_0^∞ is a domain of strong uniqueness for the operator H_p.*

Proof. Since the operator $H_p \restriction_{C_0^\infty}$ is accretive, by the Lumer-Phillips theorem it suffices to check that $\mathrm{Ran}\,(\lambda + H_p) \restriction_{C_0^\infty}$ is dense in L^p for some $\lambda > 0$, i.e. we have to show that

$$u \in L^{p'} \text{ and } \langle(\lambda + H_p)\varphi, u\rangle = 0 \quad \text{for all } \varphi \in C_0^\infty \tag{3.39}$$

yields $u = 0$.

The rest of the proof is divided into three steps.

Step 1. Let u satisfy (3.39), η, $\xi \in \mathcal{F}_R$ and $\eta = 1$ on $\operatorname{supp}\xi$. Then

$$u\xi \in \mathcal{D}(\mathcal{L}_\eta^{\frac{1}{2}}) =: \mathcal{D}.$$

Indeed, a direct computation shows that for all $\varphi \in C_0^\infty$ we have

$$\langle(\lambda + H)\varphi,\, u\xi\rangle = \langle(\lambda + H_\eta)\varphi,\, u\xi\rangle.$$

Since $\xi\varphi \in C_0^\infty$, (3.39) implies that

$$\langle(\lambda + H_\eta)\varphi,\, u\xi\rangle = \langle 2\nabla\xi \cdot \nabla\varphi + (\Delta\xi)\varphi - (\mathbf{b}\cdot\nabla\xi)\varphi,\, u\rangle. \tag{3.40}$$

It follows from the Hölder inequality that

$$\begin{aligned}
\|\nabla\xi \cdot \nabla\varphi\|_p &\le C_\xi \|\eta|\nabla\varphi|\,\|_2 = C_\xi\|\mathcal{L}_\eta^{\frac{1}{2}}\varphi\|_2,\\
\|(\Delta\xi)\varphi\|_p &\le C_\xi\|\varphi\|_2,\\
\|(\mathbf{b}\cdot\nabla\xi)\varphi\|_p &\le C_\xi\|(1+\mathcal{L}_\eta)^{\frac{1}{2}}\varphi\|_2
\end{aligned}$$

(we made use of assumption **(B1)** to derive the last estimate). Next we observe that by **(B2)** one can find a constant $C_\alpha > 0$ such that

$$\|(1+\mathcal{L}_\eta)^{\frac{1}{2}}\varphi\|_2 \le C_\alpha\|(\lambda + \operatorname{Re} H_\eta)^{\frac{1}{2}}\varphi\|_2,$$

for all $\varphi \in C_0^\infty$. Hence we conclude that

$$|\langle(\lambda + H_\eta)\varphi,\, u\xi\rangle| \le C_{u,\xi}\|(\lambda + \operatorname{Re} H_\eta)^{\frac{1}{2}}\varphi\|_2, \quad \forall\varphi \in C_0^\infty. \tag{3.41}$$

Since C_0^∞ is a core for the form $\mathcal{E}_\eta^{(b)}$, inequality (3.41) implies that the left-hand side of (3.40) defines a linear continuous functional on $\mathcal{D}$. Therefore by the Riesz representation theorem one can find a $v \in \mathcal{D}$ such that

$$\langle(\lambda + H_\eta)\varphi,\, u\xi\rangle = \langle(\lambda + \operatorname{Re} H_\eta)^{\frac{1}{2}}\varphi,\, (\lambda + \operatorname{Re} H_\eta)^{\frac{1}{2}}v\rangle. \tag{3.42}$$

As the form $\mathcal{E}_\eta^{(b)}$ is sectorial and the operator $(\lambda + \operatorname{Re} H_\eta) \ge 0$ for all $\lambda \ge 0$, it follows from Theorem 2.1.6 that

$$1 + H_\eta = (1 + \operatorname{Re} H_\eta)^{\frac{1}{2}}(\mathrm{Id} + i\,\mathcal{B})(1 + \operatorname{Re} H_\eta)^{\frac{1}{2}},$$

where $\mathcal{B}$ is a bounded self-adjoint operator on L^2. The rest of Step 1 is identical to the corresponding part of Step 1 in Theorem 3.1.4.

Step 2. Suppose that u satisfies (3.39) and $\xi \in \mathcal{F}_R$. Then there exists a constant $C = C(\alpha, p)$ which depends only on α and p such that

$$\|u\xi\|_{p'} \leq C\|u|\nabla\xi|^{\frac{2}{p'}}\|_{p'}. \tag{3.43}$$

We choose $\hat{\xi}$, $\eta \in \mathcal{F}_R$ such that $\hat{\xi} \restriction_{supp\xi} = 1$ and $\eta \restriction_{supp\hat{\xi}} = 1$. Set $\hat{u} := u\hat{\xi}$. We note that $\hat{u} \restriction_{supp\xi} = u \restriction_{supp\xi}$ and $\hat{u} \in \mathcal{D}$ by Step 1.

We introduce the functions $g_n(y) = y(|y| \wedge n)^{p'-2}$, $n \in \mathbb{N}$, $y \in \mathbb{C}$. For every $n \in \mathbb{N}$ the function g_n is clearly Lipschitz continuous, so $(p'-1)n^{2-p'}g_n$ are normal contractions. Hence, for every $v \in \mathcal{D}$ we have $g_n \circ v \in \mathcal{D}$. We set $\varphi^{(n)} = g_n \circ u\xi \in \mathcal{D}$, $n \in \mathbb{N}$.

For $n \in \mathbb{N}$ let $(\varphi_k^{(n)})_{k\in\mathbb{N}} \subset C_0^\infty$ be a sequence such that $\varphi_k^{(n)} \to \varphi^{(n)}$ in $\mathcal{D}$. Due to the choice of ξ, $\hat{\xi}$ and η we have $u = \hat{u}$ and $\eta\nabla\hat{u} = \nabla\hat{u}$ on $\operatorname{supp}\xi$. We rewrite (3.40) with $\varphi = \varphi_k^{(n)}$, make use of Step 1 and integrate by parts in the right-hand side of the obtained equality:

$$\begin{aligned} &\langle\varphi_k^{(n)}, u\xi\rangle + \langle\nabla\varphi_k^{(n)}, \nabla(u\xi)\rangle + \langle b\cdot\nabla\varphi_k^{(n)}, u\xi\rangle \\ &= \langle\nabla\varphi_k^{(n)}, u\nabla\xi\rangle - \langle\varphi_k^{(n)}, \nabla\hat{u}\cdot\nabla\xi\rangle - \langle b\,\varphi_k^{(n)}, u\nabla\xi\rangle. \end{aligned} \tag{3.44}$$

Passing to the limit as $k \to \infty$ in (3.44) and taking the real part of both sides of the obtained equality we get

$$\begin{aligned} &\left\|u\xi(|u\xi| \wedge n)^{\frac{p'-2}{2}}\right\|_2^2 + \operatorname{Re}\Big(\big\langle\nabla\big(u\xi(|u\xi| \wedge n)^{p'-2}\big), \nabla(u\xi)\big\rangle \\ &+ \big\langle b\cdot\nabla(u\xi(|u\xi| \wedge n)^{p'-2}), u\xi\big\rangle\Big) = \operatorname{Re}\Big(\big\langle\nabla\big(u\xi(|u\xi| \wedge n)^{p'-2}\big), u\nabla\xi\big\rangle \\ &- \big\langle u\xi(|u\xi| \wedge n)^{p'-2}, \nabla\hat{u}\cdot\nabla\xi\big\rangle - \big\langle b\,u\xi(|u\xi| \wedge n)^{p'-2}, u\nabla\xi\big\rangle\Big). \end{aligned} \tag{3.45}$$

We use the notation $v_n := \big(|u\xi| \wedge n\big)^{\frac{p'}{2}}$ and $w_n := |u\xi| \vee n$. Recall that $1\!\!1_n$ and $1\!\!1_{-n}$ stand for the characteristic functions of the sets $\{|u\xi| \geq n\}$ and $\{|u\xi| < n\}$ respectively. We note that $\nabla v_n = 1\!\!1_{-n}\frac{p'}{2}|u\xi|^{\frac{p'-2}{2}}\nabla|u\xi|$ and $\nabla w_n = 1\!\!1_n\nabla|u\xi|$. It follows from (3.12) that

$$\operatorname{Re}\langle\nabla\varphi^{(n)}, \nabla(u\xi)\rangle \geq \frac{4}{pp'}\|\nabla v_n\|_2^2 + n^{p'-2}\|\nabla w_n\|_2^2.$$

Next we compute

$$\begin{aligned} \operatorname{Re}\bar{u}\xi\nabla\varphi^{(n)} &= 1\!\!1_{-n}(p'-1)|u\xi|^{p'-1}\nabla|u\xi| + 1\!\!1_n)n^{p'-2}|u\xi|\nabla|u\xi| \\ &= \frac{2}{p}v_n\nabla v_n + n^{p'-2}|u\xi|\nabla w_n, \end{aligned} \tag{3.46}$$

since $\frac{2(p'-1)}{p'} = \frac{2}{p}$. Making use of (3.46), the Schwarz and Cauchy inequalities we conclude that

$$\begin{aligned}\operatorname{Re}\Big\langle b\cdot\nabla\varphi^{(n)},\, u\xi\Big\rangle &\geq \frac{\varepsilon_1}{p}\Big\|\nabla v_n\Big\|_2^2 - \frac{1}{\varepsilon_1 p}\Big\||b|v_n\Big\|_2^2\\ &- n^{p'-2}\Big(\frac{\varepsilon_2}{2}\Big\|\nabla w_n\|_2^2 + \frac{1}{2\varepsilon_2}\Big\|\mathbb{1}_n|b|w_n\Big\|_2^2\Big),\end{aligned}$$

for all $\varepsilon_1, \varepsilon_2 > 0$. Hence, applying **(B2)** and optimising in $\varepsilon_1, \varepsilon_2$, we obtain the estimate

$$\operatorname{Re}\Big\langle b\cdot\nabla(u\xi(|u\xi|\wedge n)^{p'-2}),\, u\xi\Big\rangle \geq -\frac{2\sqrt{\alpha}}{p}\Big\|\nabla v_n\Big\|_2^2 - n^{p'-2}\sqrt{\alpha}\Big\|\nabla w_n\Big\|_2^2. \tag{3.47}$$

We combine (3.12) and (3.47) and estimate the left-hand side of (3.45) below as follows.

$$\begin{aligned}&\operatorname{Re}\Big(\langle\nabla(u\xi(|u\xi|\wedge n)^{p'-2}),\, \nabla(u\xi)\rangle + \langle b\cdot\nabla(u\xi(|u\xi|\wedge n)^{p'-2}),\, u\xi\rangle\Big)\\ &+ \Big\langle |u\xi|^2(|u\xi|\wedge n)^{p'-2}\Big\rangle \geq \Big(\frac{4}{pp'} - \frac{2\sqrt{\alpha}}{p}\Big)\Big\|\nabla v_n\Big\|_2^2\\ &+ n^{p'-2}(1-\sqrt{\alpha})\Big\|\nabla w_n\Big\|_2^2 \geq \Big(\frac{4}{pp'} - \frac{2\sqrt{\alpha}}{p}\Big)\Big\|\nabla v_n\Big\|_2^2.\end{aligned} \tag{3.48}$$

In order to estimate the right-hand side of (3.45) we use the Schwarz and the Cauchy inequalities and apply **(B2)**:

$$\begin{aligned}&\Big|\Big\langle b\,u\xi(|u\xi|\wedge n)^{p'-2},\, u\nabla\xi\Big\rangle\Big| \leq \Big\langle |b||v_n|,\, \frac{|\nabla\xi|}{\xi}|u\xi|^{\frac{p'}{2}}|\Big\rangle\\ &\leq \delta\Big\|bv_n\Big\|_2^2 + \frac{1}{4\delta}\Big\|\Big(\frac{\nabla\xi}{\xi}\Big)|u\xi|^{\frac{p'}{2}}\Big\|_2^2\\ &\leq \delta\alpha\Big\|\nabla v_n\Big\|_2^2 + \frac{1}{4\delta}\Big\langle\Big(\frac{\nabla\xi}{\xi}\Big)^2|u\xi|^{p'}\Big\rangle,\end{aligned} \tag{3.49}$$

for every $\delta > 0$ (here we have used the estimate $|v^{(n)}|^2 \leq |u\xi|^{p'}$). Combining (3.15) with (3.49) we estimate the right-hand side of (3.45) above by

$$(\varepsilon + \delta\alpha)\Big\|\nabla v_n\Big\|_2^2 + \Big(\frac{1}{\varepsilon}\Big(\frac{p'-2}{p'}\Big)^2 + 1 + \frac{1}{4\delta}\Big)\Big\langle\Big(\frac{\nabla\xi}{\xi}\Big)^2|u\xi|^{p'}\Big\rangle \tag{3.50}$$

for all $\varepsilon, \delta > 0$.

Next we make use of (3.13) and (3.50), choose constants ε and δ in such way that $\varepsilon + \delta\sqrt{\alpha} = \dfrac{2}{p}(\dfrac{2}{p'} - \sqrt{\alpha}) > 0$ and obtain the estimate

$$\langle |u\xi|^2(|u\xi|\wedge n)^{p'-2}\rangle \leq C_{\alpha,p}\langle |u|^{p'}\xi^{p'-2}|\nabla\xi|^2\rangle.$$

In order to complete the proof of (3.43) we observe that by the B. Levi theorem $|u\xi|^2(|u\xi| \wedge n)^{p'-2} \to |u\xi|^{p'}$ in L^1 and $\xi^{p'-2} \leq 1$, since $0 \leq \xi \leq 1$ and $p' > 2$.

Step 3. $u = 0$.

We choose a sequence $(\xi_n)_{n\in\mathbb{N}}$ such that $\xi_n \to 1$ pointwise and $|\nabla \xi_n| \leq 1$ to see that $u\xi_n \to u$ and $u\nabla\xi_n \to 0$ in $L^{p'}$. This completes the proof. □

Next we state and prove the "localisation" theorem in the case $p = 2$.

Theorem 3.2.6. *Suppose that β, $b \in L^2_{\text{loc}}$ and conditions* **(B1)-(B2)** *hold. We also assume that the closure in L^2 of the operator $H_\eta \restriction_{C_0^\infty}$ is m-accretive. Then C_0^∞ is a domain of strong uniqueness for the operator H.*

Proof. Since the operator $H \restriction_{C_0^\infty}$ is accretive, by the Lumer-Phillips theorem it suffices to check that $\operatorname{Ran}(\lambda + H) \restriction_{C_0^\infty}$ is dense in L^2 for some $\lambda > 0$, i.e. we have to show that

$$u \in L^2 \text{ and } \langle(\lambda + H)\varphi, u\rangle = 0 \quad \text{for all} \quad \varphi \in C_0^\infty \tag{3.51}$$

yields $u = 0$.

The rest of the proof is divided into two steps.

Step 1. Let u satisfy (3.51), η, $\xi \in \mathcal{F}_R$ and $\eta = 1$ on $\operatorname{supp}\xi$. Then

$$u\xi \in \mathcal{D}(\mathcal{L}_\eta^{\frac{1}{2}}) =: \mathcal{D}.$$

The proof of this claim is identical to Step 1 of the proof of Theorem 3.2.5.

Step 2. This is similar to Step 2 of the proof of Theorem 3.1.5.

Let $\xi \in \mathcal{F}_R$ and u be as in (3.51). Our goal is to prove the following estimate:

$$\|u\xi\|_2^2 \leq C_\alpha \big\|u|\nabla\xi|\big\|_2^2 \quad \text{for all} \quad \xi \in \mathcal{F}_R,$$

where C_α is a positive constant depending on α.

Let us choose $\hat{\xi}$, $\eta \in \mathcal{F}_R$ such that $\hat{\xi} \restriction_{supp\xi} = 1$ and $\eta \restriction_{supp\hat{\xi}} = 1$. Set $\hat{u} := u\hat{\xi}$ and note that $\hat{u} \restriction_{supp\xi} = u \restriction_{supp\xi}$. By Step 1 $\hat{u} \in \mathcal{D}$.

Let $(\varphi_k)_{k\in\mathbb{N}} \subset C_0^\infty$ be a sequence such that $\varphi_k \to u\xi$ in $\mathcal{D}$. It follows from (3.40) that

$$\langle(\lambda + H_\eta)\varphi_k, u\xi\rangle = 2\langle\nabla\xi \cdot \nabla\varphi_k, u\rangle + \langle(\Delta\xi - \mathbf{b} \cdot \nabla\xi)\varphi_k, u\rangle. \tag{3.52}$$

By Step 1

$$\langle(\lambda + H_\eta)\varphi_k, u\xi\rangle = \langle\nabla\varphi_k, \nabla(u\xi)\rangle + \lambda\langle\varphi_k, u\xi\rangle + \langle b \cdot \nabla\varphi_k, u\xi\rangle.$$

Integrating by parts in the right-hand side of (3.52) and observing that $\nabla(u\varphi_k) = \varphi_k \nabla \hat{u} + u\nabla\varphi_k$ on $\operatorname{supp}\xi$ we derive the equality

$$2\langle \nabla\xi \cdot \nabla\varphi_k,\, u\rangle + \langle (\Delta\xi + \beta\cdot\nabla\xi)\varphi_k,\, u\rangle = \langle \nabla\xi\cdot\nabla\varphi_k,\, u\rangle - \langle \varphi_k\nabla\xi,\, \nabla\hat{u}\rangle.$$

Hence,

$$\begin{aligned} &\lambda\langle \varphi_k,\, u\xi\rangle + \langle \nabla\varphi_k,\, \nabla(u\xi)\rangle + \langle b\cdot\nabla\varphi_k,\, u\xi\rangle \\ &= \langle \nabla\varphi_k,\, u\nabla\xi\rangle - \langle \varphi_k,\, \nabla\hat{u}\cdot\nabla\xi\rangle - \langle (b\cdot\nabla\xi)\varphi_k,\, u\rangle. \end{aligned}$$

Passing to the limit as $k\to\infty$ in the last equality we get

$$\begin{aligned} &\|\,|\nabla(u\xi)|\,\|_2^2 + \lambda\|u\xi\|_2^2 + \langle b\cdot\nabla(u\xi),\, u\xi\rangle \\ &= \langle \nabla(u\xi),\, u\nabla\xi\rangle - \langle u\xi,\, \nabla\hat{u}\cdot\nabla\xi\rangle - \langle b\,u\xi,\, u\nabla\xi\rangle. \end{aligned} \tag{3.53}$$

We note that $\nabla(u\xi) = u\nabla\xi + \xi\nabla\hat{u}$, take the real parts of both sides of (3.53), and get

$$\lambda\|u\xi\|_2^2 + \|\,|\nabla(u\xi)|\,\|_2^2 + \operatorname{Re}\langle b\cdot\nabla(u\xi),\, u\xi\rangle = \|\,|u\nabla\xi|\,\|_2^2 - \operatorname{Re}\langle b\,u\xi,\, u\nabla\xi\rangle,$$

since $\langle u\xi,\, \nabla\hat{u}\cdot\nabla\xi\rangle = \overline{\langle (\nabla\hat{u})\,\xi,\, u\nabla\xi\rangle}$. Next we make use of assumption **(B2)** and estimate the left-hand side of the last equality below.

$$\lambda\|u\xi\|_2^2 + \|\,|\nabla(u\xi)|\,\|_2^2 + \operatorname{Re}\langle b\cdot\nabla(u\xi),\, u\xi\rangle \geq (\lambda - c(\alpha))\|u\xi\|_2^2 + (1-\sqrt{\alpha})\|\,|\nabla(u\xi)|\,\|_2^2.$$

Using the Schwarz and Cauchy inequalities and **(B2)** we conclude that

$$\begin{aligned} \left|\langle b\,u\xi,\, u\nabla\xi\rangle\right| &\leq \varepsilon\|\,|b|u\xi\|_2^2 + \frac{1}{4\varepsilon}\|\,|u\nabla\xi|\|_2^2 \\ &\leq \varepsilon\alpha\|\,|\nabla(u\xi)\|_2^2 + \varepsilon c(\alpha)\|u\xi\|_2^2 + \frac{1}{4\varepsilon}\|\,|u\nabla\xi|\|_2^2. \end{aligned} \tag{3.54}$$

Taking $\varepsilon := \dfrac{1-\sqrt{\alpha}}{\alpha} > 0$ we infer that

$$(\lambda - C_\alpha)\|u\xi\|_2^2 \leq \hat{C}_\alpha\|\,|u\nabla\xi|\,\|_2^2.$$

The last estimate implies that $u = 0$. $\square$

Before completing the proof of the uniqueness results we recall that for a fixed $\eta\in\mathcal{F}_R$ by Ω we denote the interior of $\operatorname{supp}\eta$. Since $\rho\in L^1_{\mathrm{loc}}$ we have $\int_\Omega \rho\,dx < \infty$. For the rest of the section $\langle\cdot\rangle$ stands for the integral over Ω. Let $b:\overline{\Omega}\to\mathbb{R}^d$ and $b\in C^\infty(\overline{\Omega})$. We set $\mathcal{A}_\eta := -\nabla\cdot\eta^2\nabla + b\cdot\eta^2\nabla \restriction_{C^2(\overline{\Omega})}$. Theorem 1 in [73] implies that the closure of $-\mathcal{A}_\eta$ generates a Feller semigroup on $C(\overline{\Omega})$.

Proof of Theorems 3.2.3 and 3.2.2. In order to establish the strong uniqueness of H_p we need to show that C_0^∞ is a core of the operator $H_{\eta,p}$, $\bar{r}(\alpha) < p \leq 2$. The argument goes along the same lines as Steps 3 and 4 of the proof of Theorem 3.1.9.

Let $\overline{\Omega} := \operatorname{supp} \eta$. Let $\widehat{H}_{\eta,p}$ stand for the operator in $L^p(\Omega, \rho dx) =: L^p(\Omega)$, associated with the closure of the form $\langle \eta \nabla u, \eta \nabla v \rangle + \langle b \cdot \eta^2 \nabla u, v \rangle$, $u, v \in C_0^1(\Omega)$.

Claim. The set $C^2(\overline{\Omega})$ is a domain of strong uniqueness for the operator $\widehat{H}_{\eta,p}$.

The operator $-\widehat{H}_{\eta,p}$ is the generator of a C_0-semigroup, therefore the set $(\lambda + \widehat{H}_{\eta,p})^{-1} C(\overline{\Omega})$ is a core of $\widehat{H_{\eta,p}}$ for all sufficiently large $\lambda > 0$. Hence, we need to approximate an element $(\lambda + \widehat{H}_{\eta,p})^{-1} f$, where $f \in C(\overline{\Omega})$, by functions from $C^2(\overline{\Omega})$ in the graph norm of $\widehat{H}_{\eta,p}$. Let a sequence $(\beta^{(n)})_{n \in \mathbb{N}} \subset C_0^\infty(\overline{\Omega})$ satisfy the property $\beta^{(n)} \to -\mathbf{b} 1\!\!1_\Omega$ in $L^{2p}(\Omega)$ as $n \to \infty$ (such a sequence exists since $\mathbf{b} \in L^{2p}_{\text{loc}}$). Let $\mathcal{A}_{\eta n}$ stand for the operator $\mathcal{A}_\eta$ with $b = \beta^{(n)}$. The closure of $-\mathcal{A}_{\eta n} \restriction_{C^2(\overline{\Omega})}$ generates a Feller semigroup, therefore the set $\mathcal{F}_n := (\lambda + \mathcal{A}_{\eta n}) C^2(\overline{\Omega})$ is dense in $C(\overline{\Omega})$ (and, therefore, in $L^p(\Omega)$ since the measure of Ω is finite). For every $n \in \mathbb{N}$ we take a sequence $(f_k^{(n)})_{k \in \mathbb{N}} \subset \mathcal{F}_n$ such that $f_k^{(n)} \to f$ in $C(\overline{\Omega})$ as $k \to \infty$. By $\phi_k^{(n)}$ we denote the solution of the equation $(\lambda + \mathcal{A}_{\eta n}) \phi_k^{(n)} = f_k^{(n)}$, $k \in \mathbb{N}$. Then $\phi_k^{(n)} \in C^2(\overline{\Omega}) \subset \mathcal{D}(\widehat{H}_{\eta,p}) \cap \mathcal{D}(\overline{\mathcal{A}}_{\eta n})$ and the following equalities hold in $L^p(\Omega)$:

$$
\begin{aligned}
&(\lambda + \widehat{H_{\eta,p}})^{-1} f - \phi_k^{(n)} = (\lambda + \widehat{H_{\eta,p}})^{-1} (f - f_k^{(n)}) + (\lambda + \widehat{H_{\eta,p}})^{-1} f_k^{(n)} - \phi_k^{(n)} \\
&= (\lambda + \widehat{H_{\eta,p}})^{-1} (f - f_k^{(n)}) + (\lambda + \widehat{H_{\eta,p}})^{-1} (\mathcal{A}_{\eta n} - \widehat{H_{\eta,p}})(\lambda + \bar{\mathcal{A}}_{\eta n})^{-1} f_k^{(n)} \\
&= (\lambda + \widehat{H_{\eta,p}})^{-1} (f - f_k^{(n)}) + (\lambda + \widehat{H_{\eta,p}})^{-1} (\mathbf{b} + \beta^{(n)}) \cdot \eta^2 \nabla \phi_k^{(n)}, \qquad (3.55)
\end{aligned}
$$

and

$$
f - (\lambda + \widehat{H_{\eta,p}}) \phi_k^{(n)} = f - f_k^{(n)} + (\mathbf{b} + \beta^{(n)}) \cdot \eta \nabla \phi_k^{(n)}. \qquad (3.56)
$$

Next we choose n_0 and k_0 in such way that $\| |\beta^{(n)} \eta| \|_{2p} \leq 2 (\| |\beta \eta| \|_{2p} + \| |b \eta| \|_{2p})$ for all $n \geq n_0$ and $\| f_k^{(n)} \|_\infty \leq 2 \| f \|_\infty$ for every $k \geq k_0$. We employ Theorem 3.4.1 and obtain the estimate

$$
\| \eta \nabla \phi_k^{(n)} \|_{2p} \leq C_{p,\lambda} \big(\| f \|_\infty + \| |\beta \eta| \|_{2p} + \| |b \eta| \|_{2p} \big),
$$

with C depending only on p, λ and η. Passing to the limit in k and then in n in (3.55) and (3.56) we complete the proof of the Claim.

Now we show that C_0^∞ is a domain of strong uniqueness for the operator $H_{\eta,p}$.

By the Lumer-Philips theorem strong uniqueness of $\widehat{H_{\eta,p}}$ on $C^2(\overline{\Omega})$ is equivalent to the fact that $\operatorname{Ran} (1 + \widehat{H_{\eta,p}}) \restriction_{C^2(\overline{\Omega})}$ is dense in $L^p(\Omega)$. Therefore, $\operatorname{Ran} (1 + H_{\eta,p}) \restriction_{C_b^2(\mathbb{R}^d) \cap L^p}$ is dense in L^p, i.e. $C_b^2(\mathbb{R}^d) \cap L^p$ is a core for the operator $H_{\eta,p}$.

Indeed, for any $\varepsilon > 0$ and $f \in L^p$ we can choose a function $\tilde{v}_1 \in C^2(\overline{\Omega})$ such that $\|[(1+\widehat{H_{\eta,p}})\tilde{v}_1 - f]1\!\!1_\Omega\|_p < \varepsilon/2$. Let $v \in C_b^2(\mathbb{R}^d) \cap L^p$ be an extension of $\tilde{v}_1$ and $v_2 \in C_0^2(\Omega^c)$, where Ω^c stands for the complement of Ω. The equality

$$(1+H_{\eta,p})(v_1+v_2) - f = [(1+\widehat{H_{\eta,p}})1\!\!1_\Omega v_1 - 1\!\!1_\Omega f] + [v_2 + 1\!\!1_{\Omega^c}(v_1 - f)]$$

shows that $\|(1+H_{\eta,p})v - f\|_p < \varepsilon$, provided we choose the function v_2 to satisfy the estimate $\|v_2 + 1\!\!1_{\Omega^c}(v_1 - f)\|_p < \varepsilon/2$.

In order to check that $\overline{H_\eta \restriction_{C_b^2(\mathbb{R}^d)\cap L^p}} = \overline{H_\eta \restriction_{C_b^\infty(\mathbb{R}^d)\cap L^p}}$ we use a standard approximation for functions from $C_b^2(\mathbb{R}^d)$ by elements of $C_b^\infty(\mathbb{R}^d)$.

Finally taking a sequence $(\omega_n) \subset C_0^\infty(\mathbb{R}^d)$ such that $\omega_n \to 1$ pointwise, $|\nabla\omega_n| \leq 1$ and $|\Delta\omega_n| \leq 1$ one can easily show that $\omega_n\phi \to \phi$ and $H_{\eta,p}\omega_n\phi \to H_{\eta,p}\phi$ as $n \to \infty$, i.e. $C_0^\infty(\mathbb{R}^d)$ is also a core for the operator $H_{\eta,p}$. □

Proof of Theorem 3.2.4. The proof is analogous to the previous one, namely we employ Proposition 3.4.4 in place of Theorem 3.4.1. □

3.3 Perturbations by Singular Drifts and Potentials

Next we study the problem of strong uniqueness for the generator associated with the differential expression $\mathcal{H}^{(b,q)}$.

3.3.1 Construction of Generator and Formulation of Main Results

Let $p \geq 1$. Let $q : \mathbb{R}^d \to \mathbb{C}$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ be measurable. We set $V := \operatorname{Re} q$, $W := \operatorname{Im} q$ and $\mathbf{b} := \beta - b$, where β stands for the logarithmic derivative of the measure ρdx. We assume that $V \geq 0$ a.e.

We suppose that $\beta, b, V, W \in L^p_{\text{loc}}$. Then an operator $\mathcal{H}$ given by

$$\mathcal{H}\varphi = \mathcal{H}^{(b,q)}\varphi,\ \varphi \in C_0^\infty$$

is well-defined in L^p. If the potential $|b|^2 \in PK_\alpha(\mathcal{L} \dotplus V)$ for some $0 \leq \alpha < 1$ then the operator $\mathcal{H}$ is quasi-accretive in L^p for all $p \geq r(\alpha)$, where $r(\alpha) := \dfrac{2}{2-\sqrt{\alpha}}$.

Indeed, for every $\varphi \in C_0^\infty$ we have

$$\begin{aligned}
&\operatorname{Re}\langle \mathcal{H}\varphi,\, \varphi|\varphi|^{p-2}\rangle = \langle \nabla\varphi,\, \nabla(\varphi|\varphi|^{p-2})\rangle + \langle V|\varphi|^p\rangle + \operatorname{Re}\langle b\cdot\nabla\varphi,\, \varphi|\varphi|^{p-2}\rangle \\
&\geq \frac{4(p-1)}{p^2}\|\nabla|\varphi|^{\frac{p}{2}}\|_2^2 + \langle V|\varphi|^p\rangle - \delta\|\nabla|\varphi|^{\frac{p}{2}}\|_2^2 - \frac{1}{\delta p^2}\langle |b|^2|\varphi|^p\rangle \\
&\geq \Big(\frac{4(p-1)}{p^2} - \delta - \frac{\alpha}{\delta p^2}\Big)\|\nabla|\varphi|^{\frac{p}{2}}\|_2^2 + \Big(1-\frac{\alpha}{\delta p^2}\Big)\langle V|\varphi|^p\rangle - \frac{c(\alpha)}{\delta p^2}\|\varphi\|_p^p \\
&\geq \frac{4p-2p\sqrt{\alpha}-4}{p^2}\|\nabla|\varphi|^{\frac{p}{2}}\|_2^2 + \Big(1-\frac{\sqrt{\alpha}}{p}\Big)\langle V|\varphi|^p\rangle - \frac{c(\alpha)}{\sqrt{\alpha}p}\|\varphi\|_p^p \\
&\geq -\frac{c(\alpha)}{\sqrt{\alpha}p}\|\varphi\|_p^p,
\end{aligned}$$

provided $p \geq r(\alpha)$. A similar argument shows that the sesquilinear form

$$\mathcal{E}^{(b,V)}(u,v) = \langle \nabla u,\, \nabla v\rangle + \langle b\cdot\nabla u,\, v\rangle + \langle Vu,\, v\rangle, \quad u,v \in C_0^1,$$

is sectorial. Let $\mathcal{A}$ stand for the m-sectorial operator associated (by Theorem 2.1.4) with the closure of $\mathcal{E}^{(b,V)}$. By U we denote the analytic semigroup on L^2, generated by the operator $-\mathcal{A}$. Similarly to the previous section (see Theorem 3.2.1) we construct a family of quasi-contractive semigroups U^p, $r(\alpha) \leq p < \infty$ and denote the corresponding generators by $-\mathcal{A}_p$. One can readily see that $\mathcal{A}_p \supset \mathcal{L}_p + V + b\cdot\nabla \restriction_{C_0^\infty}$.

It follows from Proposition 2.2.24 that the semigroup U^p is positive. The potential $W \in L^p_{\mathrm{loc}} \subset L^1_{\mathrm{loc}}$, and so, by Proposition 2.3.6, $|W|$ is U^p-regular. Hence by Proposition 2.3.2 the limit in L^p

$$T^p(t) = s\text{-}\lim_n \exp(-t(\mathcal{A}_p + iW_n))$$

exists for all $t \geq 0$, and T^p is a C_0-semigroup (recall that W_n, $n \geq 1$, stand for the cut-offs of W). By $-H_p$ we denote the generator of T^p. It was shown before (see the discussion after Proposition 2.3.2), that $H_p \supset \mathcal{H}$. We set $H := H_2$.

Remark. *Using the same technique one can construct the generator $-H_p$ under less restrictive conditions. Namely, let $V = V^+ - V^-$, $V^- \in PK_\gamma(\mathcal{L} \dotplus V^+)$ and $|b|^2 \in PK_\alpha(\mathcal{L} \dotplus V^+)$, with $\alpha, \gamma \in [0,1)$ satisfying $M(\alpha,\gamma) := 4 + \alpha - 4\sqrt{\alpha} - 4\gamma > 0$. Then following the same procedure we obtain a family of quasi-contractive semigroups T^p, $p_1(\alpha,\gamma) \leq p \leq p_2(\alpha,\gamma)$, where $p_1(\alpha,\gamma) := \dfrac{4}{2-\sqrt{\alpha}+\sqrt{M(\alpha,\gamma)}}$ and $p_2(\alpha,\gamma) := \dfrac{4}{2-\sqrt{\alpha}-\sqrt{M(\alpha,\gamma)}}$. However, in order to establish the strong uniqueness of H_p we have to assume that $V^- = 0$.*

Next we state the precise conditions on the coefficients of the operator $\mathcal{H}$.

(C1) There exists an $R < \infty$ such that for every $R_1 > R$ there is a constant $C = C(R_1) \geq 0$ such that for all $\varphi \in C_0^\infty$ we have

$$\left\| |\mathbf{b}|\varphi \mathbb{1}_{B_{R_1}\setminus B_R} \right\|_p^2 \leq C(\| |\nabla\varphi| \|_2^2 + \|\varphi\|_2^2),$$

where $\mathbb{1}_{B_{R_1}\setminus B_R}$ is the characteristic function of the set $B_{R_1} \setminus B_R$.

Recall that $\mathcal{F}_R$ stands for the class of spherically symmetric non-negative functions $\eta \in C_0^\infty$ such that $\eta \leq 1$ and $\eta \restriction_{B_R} = 1$. We assume that for every $\eta \in \mathcal{F}_R$

(C2) the potential $|\eta b|^2 \in PK_\alpha(\mathcal{L}_\eta)$ for some $0 \leq \alpha < 1$;

(C3) there is a constant $a(\eta) > 0$ such that $W_\eta \in PK_{a(\eta)}(\mathcal{L}_\eta \dot{+} V_\eta)$.

Assumption **(C1)** is identical to **(B1)** in the previous section. We also note that assumption **(C2)** yields $|b|^2 \in PK_\alpha(\mathcal{L})$. Conditions **(C2)** and **(C3)** imply that the form

$$\mathcal{E}_\eta^{(b,q)}(u, v) = \langle \eta\nabla u, \eta\nabla v\rangle + \langle b \cdot \eta^2 \nabla u, v\rangle + \langle q\eta u, v\rangle, \quad u, v \in C_0^1(\mathbb{R}^d),$$

is sectorial for every $\eta \in \mathcal{F}_R$.

Although it is possible to associate a C_0-semigroup with $\mathcal{H}^{b,q}$ and to carry out the localisation step under a weaker assumption that $|b|^2 \in PK_\alpha(\mathcal{L} \dot{+} V)$, we do not manage to avoid condition **(C2)** in the proof of the uniqueness results below.

Now we are ready to formulate the relevant uniqueness results.

Theorem 3.3.1. *Let β, $b \in L^4_{\text{loc}}$, $V, W \in L^2_{\text{loc}}$. We assume that conditions* **(C1)**-**(C3)** *hold. Then $C_0^\infty(\mathbb{R}^d)$ is a domain of strong uniqueness for the operator H.*

Set $\bar{r}(\alpha) := r(\alpha) \vee \frac{3}{2}$.

Theorem 3.3.2. *Let $\bar{r}(\alpha) < p < 2$. Let $\mathbf{b} \in L^{2p}_{\text{loc}}$, $\beta, V, W \in L^p_{\text{loc}}$. We assume that conditions* **(C1)**-**(C3)** *hold and $c(\alpha) = 0$ in* **(C2)**. *Then $C_0^\infty(\mathbb{R}^d)$ is a domain of strong uniqueness for the operator H_p.*

In the case $r(\alpha) < 3/2$ the uniqueness result for the operator H_p reads as follows.

Theorem 3.3.3. *Let $r(\alpha) < p \leq 3/2$. Let β, $b \in L^{\frac{2p}{2-p}}_{\text{loc}}$, $V, W \in L^p_{\text{loc}}$. We assume that conditions* **(C1)**-**(C3)** *hold and $c(\alpha) = 0$ in* **(C2)**. *Then $C_0^\infty(\mathbb{R}^d)$ is a domain of strong uniqueness for the operator H_p.*

Remark. *We suspect that the assumption $c(\alpha) = 0$ in Theorems 3.3.2 and 3.3.3 is superfluous. However, at present it is not clear how it could be avoided within the method.*

Analogously to the previous two sections the proof is carried out in steps. First the problem is reduced to the strong uniqueness for a degenerate operator on a ball in $\mathbb{R}^d$. Then the "localised" problem is investigated using the a priori estimates technique.

3.3.2 Proof of Uniqueness

For $r(\alpha) < p \leq 2$ let $-H_{\eta,p}$ stand for the generator of the C_0-semigroup on L^p, associated with the closure of the form $\mathcal{E}_\eta^{(b,q)}$. Let $H_\eta := H_{\eta,2}$.

Next we formulate the "localisation" theorems in the cases $p = 2$ and $r(\alpha) < p < 2$.

Theorem 3.3.4. *Let $\beta, b, V, W \in L^2_{\text{loc}}$. We assume that conditions* **(C1)-(C3)** *hold. We also assume that C_0^∞ is a core for the operator H_η. Then C_0^∞ is a domain of strong uniqueness for the operator H.*

Theorem 3.3.5. *Let $r(\alpha) < p < 2$. Let $\beta, b, V, W \in L^p_{\text{loc}}$. We assume that conditions* **(C1)-(C3)** *hold, with $c(\alpha) = 0$ in* **(C2)**. *We also assume that the closure in L^p of $H_{\eta,p} \restriction_{C_0^\infty}$ is m-accretive. Then C_0^∞ is a domain of strong uniqueness for the operator H_p.*

Proof of Theorem 3.3.4. We only prove the localisation result in the case $p = 2$. The proof of Theorem 3.3.5 is a simple combination of the present one and that of Theorem 3.2.5.

Since the operator $H \restriction_{C_0^\infty}$ is accretive, by the Lumer-Phillips theorem it suffices to check that $\operatorname{Ran}(\lambda + H) \restriction_{C_0^\infty}$ is dense in L^2 for some $\lambda > 0$, i.e. we have to show that

$$u \in L^2 \text{ and } \langle(\lambda + H)\varphi, u\rangle = 0 \quad \text{for all} \quad \varphi \in C_0^\infty \tag{3.57}$$

yields $u = 0$.

The rest of the proof is divided into two steps.

Step 1. Let u satisfy (3.57), η, $\xi \in \mathcal{F}_R$ and $\eta = 1$ on $\operatorname{supp} \xi$. Then

$$u\xi \in \mathcal{D}(\mathcal{L}_\eta^{\frac{1}{2}}) =: \mathcal{D}.$$

Indeed, it is easy to check that

$$\langle(\lambda + H)\varphi,\, u\xi\rangle = \langle(\lambda + H_\eta)\varphi,\, u\xi\rangle \ (\varphi \in C_0^\infty(\mathbb{R}^d)).$$

Since $\xi\varphi \in C_0^\infty(\mathbb{R}^d)$, (3.57) implies that

$$\langle(\lambda + H)\varphi,\, u\xi\rangle = 2\langle\nabla\xi \cdot \nabla\varphi,\, u\rangle + \langle(\Delta\xi - \mathbf{b}\cdot\nabla\xi)\varphi,\, u\rangle. \tag{3.58}$$

Applying the Schwarz inequality and **(C1)** we have

$$\begin{aligned}\|\nabla\xi \cdot \nabla\varphi\|_2 &\le C_\xi \|\eta|\nabla\varphi|\,\|_2 \le \|(1 + \mathcal{L}_\eta)^{\frac{1}{2}})\varphi\|_2,\\ \|(\Delta\xi)\varphi\|_2 &\le C_\xi\|\varphi\|_2,\\ \|(\beta\cdot\nabla\xi)\varphi\|_2 &\le C_\xi\|(1 + \mathcal{L}_\eta)^{\frac{1}{2}})\varphi\|_2,\end{aligned}$$

Observing that by **(C2)** one can find a constant $C = C(\alpha, \eta) > 0$ such that

$$\|(1 + \mathcal{L}_\eta)^{\frac{1}{2}}\varphi\|_2 \le C\|(\lambda + \operatorname{Re} H_\eta)^{\frac{1}{2}}\varphi\|_2,$$

we conclude that

$$|\langle(\lambda + H_\eta)\varphi,\, u\xi\rangle| \le C_{u,\xi}\|(\lambda + \operatorname{Re} H_\eta)^{\frac{1}{2}}\varphi\|_2. \tag{3.59}$$

Estimate (3.59) implies that the left-hand side of (3.58) defines a linear bounded functional on $\mathcal{D}$, i.e. there is an element $v \in \mathcal{D}$ such that

$$\langle(\lambda + H_\eta)\varphi,\, u\xi\rangle = \langle(\lambda + \operatorname{Re} H_\eta)^{\frac{1}{2}}\varphi,\, (\lambda + \operatorname{Re} H_\eta)^{\frac{1}{2}}v\rangle. \tag{3.60}$$

Since the form $\mathcal{E}_\eta^{(b,q)}$ is sectorial and the operator $\lambda + \operatorname{Re} H_\eta \ge 0$ for all $\lambda \ge c(\alpha)$ it follows from Theorem 2.1.7 that

$$\lambda + H_\eta = (\lambda + \operatorname{Re} H_\eta)^{\frac{1}{2}}(\mathrm{Id} + i\,\mathcal{B})(\lambda + \operatorname{Re} H_\eta)^{\frac{1}{2}},$$

where $\mathcal{B}$ is a bounded self-adjoint operator in L^2. The operator $\mathrm{Id} - i\,\mathcal{B} : L^2 \to L^2$ is clearly bijective, and the mapping $(\lambda + \operatorname{Re} H_\eta)^{\frac{1}{2}} : \mathcal{D} \to L^2$ is known to be isomorphic. Therefore for every $v \in \mathcal{D}$ one can find a (unique) $w \in \mathcal{D}$ such that

$$(\mathrm{Id} - i\,\mathcal{B})(\lambda + \operatorname{Re} H_\eta)^{\frac{1}{2}}w = (\lambda + \operatorname{Re} H_\eta)^{\frac{1}{2}}v. \tag{3.61}$$

Combining (3.60) and (3.61) we get

$$\langle(\lambda + H_\eta)\varphi,\, u\xi\rangle = \langle(\mathrm{Id} + i\,\mathcal{B})(\lambda + \operatorname{Re} H_\eta)^{\frac{1}{2}}\varphi,\, (\lambda + \operatorname{Re} H_\eta)^{\frac{1}{2}}w\rangle.$$

We employ the strong uniqueness of $H_\eta \restriction_{C_0^\infty}$ and obtain the equality $\langle \psi, u\xi \rangle = \langle \psi, w \rangle$ for all $\psi \in L^2$. Hence, $w = u\xi$ μ-a.e.

Step 2. Let $\xi \in \mathcal{F}_R$ and u be as in (3.51). Our goal is to prove the following estimate:

$$\left\|u\xi\right\|_2^2 \leq C_\alpha \left\|u|\nabla\xi|\right\|_2^2 \text{ for all } \xi \in \mathcal{F}_R,$$

where C_α is a positive constant dependent on α.

We choose $\hat{\xi}$, $\eta \in \mathcal{F}_R$ such that $\hat{\xi} \restriction_{supp\xi} = 1$ and $\eta \restriction_{supp\hat{\xi}} = 1$. Set $\hat{u} := u\hat{\xi}$ and note that $\hat{u} \restriction_{supp\xi} = u \restriction_{supp\xi}$. By Step 1 $\hat{u} \in \mathcal{D}$.

Let $(\varphi_k)_{k\in\mathbb{N}} \subset C_0^\infty$ be a sequence such that $\varphi_k \to u\xi$ in $\mathcal{D}$. It follows from (3.58) that

$$\langle (\lambda + H_\eta)\varphi_k, u\xi \rangle = 2\langle \nabla\xi \cdot \nabla\varphi_k, u \rangle + \langle (\Delta\xi - \mathbf{b} \cdot \nabla\xi)\varphi_k, u \rangle. \tag{3.62}$$

By Step 1

$$\langle (\lambda + H_\eta)\varphi_k, u\xi \rangle = \langle \nabla\varphi_k, \nabla(u\xi) \rangle + \lambda\langle \varphi_k, u\xi \rangle + \langle b \cdot \nabla\varphi_k, u\xi \rangle + \langle q\varphi_k, u\xi \rangle.$$

Integrating by parts in the right-hand side of (3.62) and observing that $\nabla(u\varphi_k) = \varphi_k \nabla\hat{u} + u\nabla\varphi_k$ on supp$\,\xi$ we derive the equality

$$2\langle \nabla\xi \cdot \nabla\varphi_k, u \rangle + \langle (\Delta\xi + \beta \cdot \nabla\xi)\varphi_k, u \rangle = \langle \nabla\xi \cdot \nabla\varphi_k, u \rangle - \langle \varphi_k \nabla\xi, \nabla\hat{u} \rangle.$$

Hence,

$$\begin{aligned} &\lambda\langle \varphi_k, u\xi \rangle + \langle \nabla\varphi_k, \nabla(u\xi) \rangle + \langle b \cdot \nabla\varphi_k, u\xi \rangle + \langle q\varphi_k, u\xi \rangle \\ &= \langle \nabla\varphi_k, u\nabla\xi \rangle - \langle \varphi_k, \nabla\hat{u} \cdot \nabla\xi \rangle - \langle (b \cdot \nabla\xi)\varphi_k, u \rangle. \end{aligned}$$

Passing to the limit as $k \to \infty$ in the last equality we get

$$\begin{aligned} &\|\,|\nabla(u\xi)|\,\|_2^2 + \lambda\|u\xi\|_2^2 + \langle b \cdot \nabla(u\xi), u\xi \rangle + \langle q|u\xi|^2 \rangle \\ &= \langle \nabla(u\xi), u\nabla\xi \rangle - \langle u\xi, \nabla\hat{u} \cdot \nabla\xi \rangle - \langle b\,u\xi, u\nabla\xi \rangle. \end{aligned} \tag{3.63}$$

We note that $\nabla(u\xi) = u\nabla\xi + \xi\nabla\hat{u}$, take the real parts of both sides of (3.63), and get

$$\lambda\|u\xi\|_2^2 + \|\,|\nabla(u\xi)|\,\|_2^2 + \mathrm{Re}\,\langle b \cdot \nabla(u\xi), u\xi \rangle + \langle V|u\xi|^2 \rangle = \|\,|u\nabla\xi|\,\|_2^2 - \mathrm{Re}\,\langle b\,u\xi, u\nabla\xi \rangle,$$

since $\langle u\xi, \nabla\hat{u} \cdot \nabla\xi \rangle = \overline{\langle (\nabla\hat{u})\,\xi, u\nabla\xi \rangle}$. Next we make use of assumption **(C2)** and conclude that

$$\begin{aligned} &\lambda\|u\xi\|_2^2 + \|\,|\nabla(u\xi)|\,\|_2^2 + \mathrm{Re}\,\langle b \cdot \nabla(u\xi), u\xi \rangle + \langle V|u\xi|^2 \rangle \\ &\geq (\lambda - c(\alpha))\|u\xi\|_2^2 + (1 - \sqrt{\alpha})\left(\|\,|\nabla(u\xi)|\,\|_2^2 + \langle V|u\xi|^2 \rangle\right). \end{aligned}$$

in order to complete the estimate we make use of the Schwarz and Cauchy inequalities and assumption **(C2)**:

$$\begin{aligned}\big|\langle b\,u\xi,\, u\nabla\xi\rangle\big| &\le \varepsilon\|\,|b|u\xi\|_2^2 + \frac{1}{4\varepsilon}\|\,|u\nabla\xi|\|_2^2\\ &\le \varepsilon\alpha\|\,|\nabla(u\xi)|\|_2^2 + \varepsilon c(\alpha)\|u\xi\|_2^2 + \frac{1}{4\varepsilon}\|\,|u\nabla\xi|\|_2^2.\end{aligned}$$

Taking $\varepsilon := \dfrac{1-\sqrt{\alpha}}{\alpha} > 0$ we infer that

$$(\lambda - C_\alpha)\|u\xi\|_2^2 \le \hat{C}_\alpha\|\,|u\nabla\xi|\,\|_2^2.$$

The last estimate implies that $u = 0$. □

Recall that for $\eta \in \mathcal{F}_R$ we set $\Omega := \operatorname{Int}(\operatorname{supp}\eta)$. Till the end of this section $\langle\cdot\rangle$ stands for the integral over Ω.

Let $\eta \in \mathcal{F}_R$. By $-\widehat{\mathcal{L}}_{\eta,b,p}$ and $-\widehat{H}_{\eta,p}$ we denote the generators of C_0-semigroups on $L^p(\Omega)$, associated with the closures of the forms

$$\langle\eta\nabla u,\, \eta\nabla v\rangle + \langle b\cdot\eta^2\nabla u,\, v\rangle, \quad u, v \in C_0^1(\Omega),$$

and

$$\langle\eta\nabla u,\, \eta\nabla v\rangle + \langle b\cdot\eta^2\nabla u,\, v\rangle + \langle q\eta\, u,\, v\rangle, \quad u, v \in C_0^1(\Omega),$$

respectively.

Proof of Theorems 3.3.1 and 3.3.2. Let $\bar{r}(\alpha) < p \le 2$. It remains to show that C_0^∞ is a core of the operator $H_{\eta,p}$. We break up the proof in two steps. Here we only discuss the first step since the second one is almost identical to the proofs of Theorems 3.2.3 and 3.2.2.

Step 1. We are going to show that the set $\bigcup_{m\in\mathbb{N}}(m+\widehat{\mathcal{L}}_{\eta,b,p})^{-1}C(\overline{\Omega})$ is a domain of strong uniqueness for $\widehat{H}_{\eta,p}$.

The operator $-\widehat{H}_{\eta,p}$ is the generator of a C_0-semigroup, therefore the set $\mathcal{D}_1$: $(\lambda+\widehat{H}_{\eta,p})^{-1}L^\infty(\Omega)$ is a core for $H_{\eta,p}$ for all $\lambda > c(\alpha)$. It follows from Lemma 2.3.4 that for every $\varphi \in \mathcal{D}_1$ we have $\widehat{H}_{\eta,p}\varphi = \widehat{\mathcal{L}}_{\eta,b,p}\varphi + (V_\eta + iW_\eta)\varphi$. Let $f \in L^\infty(\Omega)$. Then the following equalities hold.

$$\begin{aligned}&s\text{-}L^p\text{-}\lim_m m(m+\widehat{\mathcal{L}}_{\eta,b,p})^{-1}(\lambda+\widehat{H}_{\eta,p})^{-1}f = (\lambda+\widehat{H}_{\eta,p})^{-1}f,\\ &s\text{-}L^p\text{-}\lim_m m\widehat{\mathcal{L}}_{\eta,b,p}(m+\widehat{\mathcal{L}}_{\eta,b,p})^{-1}(\lambda+\widehat{H}_{\eta,p})^{-1}f = \mathcal{L}_{\eta,b,p}(\lambda+\widehat{H}_{\eta,p})^{-1}f\\ &s\text{-}L^p\text{-}\lim_m m(V_\eta+iW_\eta)(m+\widehat{\mathcal{L}}_{\eta,b,p})^{-1}(\lambda+\widehat{H}_{\eta,p})^{-1}f\\ &= (V_\eta+iW_\eta)(\lambda+\widehat{H}_{\eta,p})^{-1}f.\end{aligned}$$

Thus the set $\bigcup_{m\in\mathbb{N}}(m+\widehat{\mathcal{L}}_{\eta,b,p})^{-1}(\lambda+\widehat{H}_{\eta,p})^{-1}L^\infty(\Omega)$ is a core for the operator $\widehat{H}_{\eta,p}$. Therefore so is the set $\bigcup_{m\in\mathbb{N}}(m+\widehat{\mathcal{L}}_{\eta,b,p})^{-1}L^\infty(\Omega)$.

For any $f \in L^\infty(\Omega)$ one can find a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset C(\overline{\Omega})$ such that $\sup_k \|\varphi_k\|_\infty < \infty$ and $\|\varphi_k - f\|_p \to 0$ as $k \to \infty$. The operators $\widehat{\mathcal{L}}_{\eta,b,p}(m+\widehat{\mathcal{L}}_{\eta,b,p})^{-1}$, $m \in \mathbb{N}$ are clearly bounded in $L^p(\Omega)$ for every $1 \le p < \infty$. Therefore $\bigcup_{m\in\mathbb{N}}(m+\widehat{\mathcal{L}}_{\eta,b,p})^{-1}C(\overline{\Omega})$ is a core for the operator $\widehat{H}_{\eta,p}$. This completes the proof of Step 1.

Step 2. The rest of the proof is similar to the proof of Theorems 3.2.3 and 3.2.2. □

3.4 A Priori Estimates

Now we are heading towards establishing a priori estimates for the gradients of the solutions of the equation

$$\lambda u - (\nabla + b)\cdot \eta^2 \nabla u = f \tag{3.64}$$

on the ball $\overline{\Omega}$ with smooth b and continuous f.

Theorem 3.4.1. *Let $p > 3/2$ and $\beta \in L^{2p}_{\mathrm{loc}}$. Let u be the solution of* (3.64) *with $b \in C^\infty(\overline{\Omega})$, $\lambda > 0$ and $f \in (\lambda + \mathcal{A}_\eta)C^2(\overline{\Omega})$. Set $K_p := \|\,|\beta\eta| + |b\eta| + |\nabla\eta|\,\|_{2p}$. Then there exists a constant C_p depending only on p such that*

$$\|\eta\nabla u\|_{2p} \le C_p\Big(1+\frac{1}{\sqrt{\lambda}}\Big)\Big(\sqrt{\|f\|_\infty\|f\|_p} + \|f\|_\infty K_p\Big).$$

In order to prove Theorem 3.4.1 we need the following two auxiliary results.

Lemma 3.4.2. *Let $\beta \in L^{2p}_{\mathrm{loc}}$. Let u be the solution of* (3.64). *Then*

$$\|u\|_p \le C_p\Big(\frac{\|f\|_p}{\lambda} + \frac{\|f\|_\infty}{\lambda^2}\|\,|(b-\beta)|\eta\|_{2p}^2\Big), \tag{3.65}$$

and

$$\|u\|_2 \le C_p\Big(\frac{\|f\|_2}{\lambda} + \frac{\|f\|_\infty}{\lambda^{\frac{p+2}{2}}}\|\,|(b-\beta)|\eta\|_{2p}^p\Big). \tag{3.66}$$

Proof. It follows from (3.64) that

$$\lambda\|u\|_p^p - \langle(\nabla+\beta)\cdot\eta^2\nabla u,\, u|u|^{p-2}\rangle = \langle(b-\beta)\cdot\eta^2\nabla u,\, u|u|^{p-2}\rangle + \langle f,\, u|u|^{p-2}\rangle.$$

Integrating by parts and using Theorem 2.2.25 we obtain the inequality

$$\begin{aligned}
&\lambda\|u\|_p^p + \frac{4}{pp'}\|\eta|\nabla(u|u|^{\frac{p-2}{2}})|\,\|_2^2 \\
&\le \operatorname{Re}\left(\frac{2}{p}\left\langle(b-\beta)\cdot\eta^2\nabla(u|u|^{\frac{p-2}{2}}),\, u|u|^{\frac{p-2}{2}}\right\rangle + \langle f,\, u|u|^{p-2}\rangle\right).
\end{aligned} \tag{3.67}$$

Making use of the Hölder and Young inequalities we estimate the right-hand side of (3.67) as follows:

$$|\langle f,\, u|u|^{p-2}\rangle| \le \frac{\lambda}{3}\|u\|_p^p + C_p\lambda^{1-p}\|f\|_p^p$$

and

$$\begin{aligned}
&\left|\frac{2}{p}\langle(b-\beta)\cdot\eta^2\nabla(u|u|^{\frac{p-2}{2}}),\, u|u|^{\frac{p-2}{2}}\rangle\right| \\
&\le \frac{2}{p}\|\,|\eta\nabla(u|u|^{\frac{p-2}{2}})|\,\|_2\|u\|_p^{\frac{p-1}{2}}\|u\|_\infty^{\frac{1}{2}}\|\,|(b-\beta)\eta|\,\|_{2p} \\
&\le \frac{4}{pp'}\|\,\eta\nabla(u|u|^{\frac{p-2}{2}})\|_2^2 + \frac{\lambda}{3}\|u\|_p^p + C_p\lambda^{1-2p}\|f\|_\infty^p\|\,|(b-\beta)\eta|\,\|_{2p}^{2p}
\end{aligned}$$

Applying the last two estimates to (3.67) we obtain (3.65). In order to derive (3.66) we use (3.67) with $p=2$ and employ the estimate

$$\begin{aligned}
|\langle(b-\beta)\cdot\eta^2\nabla u,\, u\rangle| &\le \langle|u|^{\frac{1}{p}}|b-\beta|\eta,\, \eta|\nabla u||u|^{\frac{1}{p'}}\rangle \\
&\le \|\eta|\nabla u|\|_2^2 + \frac{\lambda}{3}\|u\|_2^2 + C_p\lambda^{-1-p}\|\,|b-\beta|\eta\|_{2p}^{2p}.
\end{aligned}$$

□

Let $w \in C^2(\overline{\Omega})$. We set $|\eta\nabla w|_\varepsilon^2 := |\eta\nabla w|^2 + \varepsilon^2$ with $\varepsilon = 0$ if $p \ge 2$, and $\varepsilon > 0$ otherwise. Set $\chi := |\eta\nabla w|_\varepsilon^{p-2}$.

Lemma 3.4.3. *If $p > 3/2$ then*

$$\begin{aligned}
\Big\|\,|\eta\nabla w|_\varepsilon^{p-1}|\eta\nabla w|\,\Big\|_2^2 &\le \varepsilon^{2p-2}\Big\|w\Big\|_2^2 + C_p\Big(\varepsilon^2 + \|w\|_\infty^2\Big)\Big\|\chi\nabla\cdot\eta^2\nabla w\Big\|_2^2 \\
&\quad + C_p\Big(\varepsilon^2 + \|w\|_\infty^2\Big)\Big\|\chi|\eta\nabla w|(|\beta|\eta + |\nabla\eta|)\Big\|_2^2,
\end{aligned}$$

where $\nabla\cdot\eta^2\nabla w := \sum_k \nabla_k(\eta^2\nabla_k w)$.

Proof. We break up the proof into several steps.

Step 1. We claim that

$$\begin{aligned}
\Big\|\,|\eta\nabla w|_\varepsilon^{p-1}|\eta\nabla w|\,\Big\|_2^2 &\le \varepsilon^{2p-2}\Big\|w\Big\|_2^2 + 2\Big(p-1\Big)^2\Big\|w\Big\|_\infty^2\Big\|\,|\eta\nabla w|_\varepsilon^{p-3}|\nabla|\eta\nabla w|^2\,\Big\|_2^2 \\
&\quad + 2\Big(\varepsilon^2 + \|w\|_\infty^2\Big)\Big(\|\chi\nabla\cdot\eta^2\nabla w\|_2^2 + \|\chi|\eta\nabla w||\beta\eta|\,\|_2^2\Big).
\end{aligned}$$

Indeed, integrating by parts we get

$$\Big\| \, |\eta\nabla w|_\varepsilon^{p-1}|\eta\nabla w| \, \Big\|_2^2 = -\Big\langle w|\eta\nabla w|_\varepsilon^{2p-2}, \, (\nabla+\beta)\cdot(\eta^2\nabla w)\Big\rangle$$
$$- \Big(p-1\Big)\Big\langle w|\eta\nabla w|_\varepsilon^{2p-4}, \eta^2\nabla w\cdot\nabla|\eta\nabla w|_\varepsilon^2\Big\rangle. \qquad (3.68)$$

The equality $\nabla|\eta\nabla w|_\varepsilon^2 = \nabla|\eta\nabla w|^2$ and the Schwarz inequality imply that the last term in the right-hand side of (3.68) is estimated above by

$$(p-1)\|w\|_\infty\langle|\eta\nabla w|_\varepsilon^{p-1}|\eta\nabla w|, \, |\eta\nabla w|_\varepsilon^{p-3}|\eta\nabla|\eta\nabla w|^2|\rangle$$
$$\le \frac{1}{4}\| \, |\eta\nabla w|_\varepsilon^{p-1}|\eta\nabla w| \, \|_2^2 + (p-1)^2\|w\|_\infty^2\| \, |\eta\nabla w|_\varepsilon^{p-3}|\eta\nabla|\eta\nabla w|^2| \, \|_2^2. \qquad (3.69)$$

Using the definition and applying the Schwarz and Young inequalities to the first term in the right-hand side of (3.68) we derive the estimate

$$|\langle w|\eta\nabla w|_\varepsilon^{2p-2}, \, (\nabla+\beta)\cdot\eta^2\nabla w\rangle|$$
$$\le \langle(\varepsilon^2+|\eta\nabla w|^2)\chi^2|w|, \, |\nabla\cdot\eta^2\nabla w| + |\eta\nabla w||\beta\eta|\rangle \le \frac{1}{4}\|\chi|\eta\nabla w|^2\|_2^2$$
$$+\frac{\varepsilon^2}{4}\|\chi w\|_2^2 + (\varepsilon^2+\|w\|_\infty^2)\left(\|\chi\nabla\cdot\eta^2\nabla w\|_2^2 + \|\chi|\eta\nabla w||\beta\eta| \, \|_2^2\right). \qquad (3.70)$$

Combining (3.68)-(3.70) we obtain

$$\Big\| \, |\eta\nabla w|_\varepsilon^{p-1}|\eta\nabla w| \, \Big\|_2^2 \le \frac{\varepsilon^2}{4}\Big\|\chi w\Big\|_2^2 + \frac{1}{4}\Big\|\chi|\eta\nabla w|^2\Big\|_2^2 + \frac{1}{4}\Big\| \, |\eta\nabla w|_\varepsilon^{p-1}|\eta\nabla w| \, \Big\|_2^2$$
$$+ \Big(p-1\Big)^2\Big\|w\Big\|_\infty^2\Big\| \, |\eta\nabla w|_\varepsilon^{p-3}|\eta\nabla|\eta\nabla w|^2| \, \Big\|_2^2$$
$$+ \Big(\varepsilon^2+\|w\|_\infty^2\Big)\Big(\|\chi\nabla\cdot\eta^2\nabla w\|_2^2 + \|\chi|\eta\nabla w||\beta\eta| \, \|_2^2\Big).$$

Next we note that $\chi \le \varepsilon^{p-2}$ when $p<2$. Therefore $\varepsilon^2\chi^2 \le \varepsilon^{2p-2}$ for all $p\ge 1$ (since $\varepsilon=0$ when $p\ge 2$). Hence, $\varepsilon^2\|\chi^2 w\|_2^2 \le \varepsilon^{2p-2}\|w\|_2^2$. Further, it is easy to see that $\chi|\eta\nabla w|^2 \le |\eta\nabla w|_\varepsilon^{p-1}|\eta\nabla w|$ for $p\ge 1$. This completes the proof of the claim.

Step 2. Let $|\cdot|_{HS}$ stand for the Hilbert-Schmidt norm of an operator in $\mathbb{C}^d$. We introduce the quantities

$$I_\varepsilon := \|\chi\eta^2|D^2w|_{HS}\|_2 \text{ and } J_\varepsilon := \| \, |\eta\nabla w|_\varepsilon^{p-3}|\eta\nabla|\eta\nabla w|^2| \, \|_2$$

and set $\nabla_k := \dfrac{\partial}{\partial x_k}$ and $\nabla_{jk}^2 := \dfrac{\partial^2}{\partial x_k\partial x_j}$, $k,j=1,\dots,d$. The following estimate holds.

$$\Big|I_\varepsilon^2 + (p/2-1)J_\varepsilon^2 - \|\chi\nabla\cdot\eta^2\nabla w\|_2^2\Big| \le 2I_\varepsilon\Big\|\chi|\eta\nabla w|(|\beta\eta|+|\nabla\eta|)\Big\|_2$$
$$+3\Big|p-2\Big|\Big\|\chi|\eta\nabla|\eta\nabla w|\Big\|_2\Big(\|\chi|\eta\nabla w||\nabla\eta| \, \|_2 + \|\chi\nabla\cdot\eta^2\nabla w\|_2\Big)$$
$$+\Big\|\chi\nabla\cdot\eta^2\nabla w\Big\|_2\Big\|\chi|\eta\nabla w||\beta\eta|\Big\|_2 + 2\Big\|\chi|\eta\nabla w|(|\beta\eta|+|\nabla\eta|)\Big\|_2^2.$$

Indeed, integrating by parts twice we get

$$\begin{aligned}
&\langle \nabla \cdot \eta^2 \nabla w,\, (\nabla + \beta) \cdot \chi^2 \eta^2 \nabla w \rangle = \sum_{k,j} \langle \nabla_k (\eta^2 \nabla_k w),\, (\nabla_j + \beta_j)(\chi^2 \eta^2 \nabla_j w) \rangle \\
&= \sum_{k,j} -\langle \nabla_j \nabla_k (\eta^2 \nabla_k w),\, \chi^2 \eta^2 \nabla_j w \rangle = \sum_{k,j} \langle \nabla_j (\eta^2 \nabla_k w),\, (\nabla_k + \beta_k)(\chi^2 \eta^2 \nabla_j w) \rangle \\
&= \sum_{k,j} \Big[\langle \nabla_j (\eta^2 \nabla_k w),\, \chi^2 \nabla_k (\eta^2 \nabla_j w) \rangle + \langle \nabla_j (\eta^2 \nabla_k w),\, \eta^2 \nabla_j w \nabla_k \chi^2 \rangle \\
&+ \langle \nabla_j (\eta^2 \nabla_k w),\, \beta_k \chi^2 \eta^2 \nabla_j w \rangle \Big]. \qquad (3.71)
\end{aligned}$$

A straightforward computation gives

$$\begin{aligned}
&\sum_{k,j} \langle \nabla_j (\eta^2 \nabla_k w),\, \chi^2 \nabla_k (\eta^2 \nabla_j w) \rangle \\
&= I_\varepsilon^2 + 4 \sum_{k,j} \langle \nabla_j (\eta^2 \nabla_k w),\, \chi^2 \eta \nabla_j w \nabla_k \eta \rangle + 4 \| \chi \eta \nabla w \cdot \nabla \eta \|_2^2.
\end{aligned}$$

Hence, by the Schwarz inequality

$$\begin{aligned}
&\Big| \sum_{k,j} \langle \nabla_j (\eta^2 \nabla_k w),\, \chi^2 \nabla_k (\eta^2 \nabla_j w) \rangle - I_\varepsilon^2 \Big| \\
&\le 4 I_\varepsilon \| \chi |\eta \nabla w| |\nabla \eta| \,\|_2 + 4 \| \chi |\eta \nabla w| |\nabla \eta| \,\|_2^2. \qquad (3.72)
\end{aligned}$$

Further, we observe that

$$\nabla |\eta \nabla w|^2 = 2\eta^2 D^2 w \nabla w + 2\eta \nabla \eta |\nabla w|^2$$

and

$$\nabla \chi^2 = (p-2) |\eta \nabla w|_\varepsilon^{2p-6} \nabla |\eta \nabla w|^2.$$

Thus

$$\begin{aligned}
&\sum_{j,k} \langle \nabla_j (\eta^2 \nabla_k w),\, \eta^2 \nabla_j w \nabla_k \chi^2 \rangle = \sum_{k,j} \langle \eta^2 (\eta^2 \nabla_{kj}^2 w + 2\eta \nabla_j \eta \nabla_k w),\, \nabla_j w \nabla_k \chi^2 \rangle \\
&= \frac{p-2}{2} J_\varepsilon^2 + \sum_{k,j} \langle \eta \nabla_j \eta \nabla_k w,\, \eta^2 \nabla_j w \nabla_k \chi^2 \rangle.
\end{aligned}$$

Therefore the Schwarz inequality implies that

$$\begin{aligned}
&\Big| \sum_{k,j} \langle \nabla_j (\eta^2 \nabla_k w),\, \eta^2 \nabla_j w \nabla_k \chi^2 \rangle - \frac{p-2}{2} J_\varepsilon^2 \Big| \\
&\le 2 \big| p-2 \big| \Big\| \chi |\eta \nabla |\eta \nabla w|| \Big\|_2 \Big\| \chi |\eta \nabla w| |\nabla \eta| \Big\|_2. \qquad (3.73)
\end{aligned}$$

Finally, it follows from the Schwarz and Cauchy inequalities that

$$\begin{aligned}&\Big|\sum_{k,j}\langle\nabla_j(\eta^2\nabla_k w),\ \beta_k\chi^2\eta^2\nabla_j w\rangle\Big|\\&\le I_\varepsilon\|\chi|\eta\nabla w||\beta\eta|\,\|_2+\|\chi|\eta\nabla w||\beta\eta|\,\|_2^2+\|\chi|\eta\nabla w||\nabla\eta|\,\|_2^2.\end{aligned}\tag{3.74}$$

Next we return to (3.71) and consider the expression $(\nabla+\beta)\cdot\chi^2\eta^2\nabla w$. A straightforward computation shows that

$$\begin{aligned}(\nabla+\beta)\cdot\chi^2\eta^2\nabla w =&\chi^2\nabla\cdot\eta^2\nabla w+\chi^2\beta\cdot\eta^2\nabla w\\&+(2p-4)\chi|\eta\nabla w|_\varepsilon^{p-4}|\eta\nabla w|\nabla|\eta\nabla w|\cdot\eta^2\nabla w.\end{aligned}$$

We observe that $|\eta\nabla w|_\varepsilon^{p-4}|\eta\nabla w|^2\le\chi$ (since $|\eta\nabla w|\le|\eta\nabla w|_\varepsilon$) and make use of the Hölder inequality to conclude that

$$\begin{aligned}&\Big|\langle\nabla\cdot\eta^2\nabla w,\ (\nabla+\beta)\cdot\chi^2\eta^2\nabla w\rangle-\|\chi\nabla\cdot\eta^2\nabla w\|_2^2\Big|\\&\le\|\chi\nabla\cdot\eta^2\nabla w\|_2\|\chi|\eta\nabla w||\beta\eta|\,\|_2+|2p-4|\|\chi\nabla\cdot\eta^2\nabla w\|_2\|\chi|\eta\nabla|\eta\nabla w||\,\|_2.\end{aligned}\tag{3.75}$$

Combining (3.71)-(3.75) we complete the proof of Step 2.

The Schwarz inequality implies that

$$|\eta\nabla|\eta\nabla w|^2|\le|\eta\nabla w|(2\eta^2|D^2w|_{HS}+|\nabla\eta||\eta\nabla w|).$$

Therefore for every $\delta>1$ there is a constant $C_\delta>0$ such that

$$J_\varepsilon^2\le 4\|\,|\eta\nabla w|_\varepsilon^{p-3}|\eta\nabla|\eta\nabla w|\,|\,\|_2^2\le 4\delta I_\varepsilon^2+C_\delta\|\chi|\eta\nabla w||\nabla\eta|\,\|_2^2.\tag{3.76}$$

Set $r_p:=min(0,p-2)$. Due to Step 2 there exists a constant $C_{p,\delta}>0$ such that

$$\begin{aligned}(1+2\delta r_p)I_\varepsilon^2&\le C_{p,\delta}\|\chi\nabla\cdot\eta^2\nabla w\|_2^2+C_{p,\delta}\|\chi|\eta\nabla w|(|\beta\eta|+|\nabla\eta|)\|_2^2\\&\le C_{p,\delta}I_\varepsilon(\|\chi\nabla\cdot\eta^2\nabla w\|_2+\|\chi|\eta\nabla w|(|\beta\eta|+|\nabla\eta|)\|_2).\end{aligned}\tag{3.77}$$

If $p>3/2$ then we can find such a $\delta>1$ that $1+2\delta r_p>0$. Employing the Cauchy inequality we conclude that

$$I_\varepsilon^2\le C_p\|\chi\nabla\cdot\eta^2\nabla w\|_2^2+C_p\|\chi|\eta\nabla w|(|\beta\eta|+|\nabla\eta|)\|_2^2.\tag{3.78}$$

We substitute (3.78) in (3.76) and make use of Step 1. This completes the proof of the lemma. □

Proof of Theorem 3.4.1. Let first $p \geq 2$. Equation (3.64) yields

$$\|\chi\nabla\cdot\eta^2\nabla u\|_2 \leq \lambda\|\chi u\|_2 + \|\chi f\|_2 + \|\chi|\eta\nabla u||b\eta|\,\|_2. \qquad (3.79)$$

Combining Lemma 3.4.3 with (3.79) and applying the Hölder inequality we get the estimate

$$\|\eta|\nabla u|\,\|_{2p}^{2p} \leq C_p\|f\|_\infty^2\|\eta|\nabla u|\,\|_{2p}^{2p-4}(\lambda^2\|u\|_p^2 + \|f\|_p^2 + \|\eta|\nabla u|\,\|_{2p}^2 K_p^2).$$

By the Young inequality we have

$$\|\eta|\nabla u|\|_{2p}^4 \leq C_p\|f\|_\infty^2(\lambda^2\|u\|_p^2 + \|f\|_p^2) + C_p\|f\|_\infty^4 K_p^4.$$

The statement of the theorem now follows from (3.65).

Let now $3/2 < p < 2$. Making successive use of (3.64), the Schwarz inequality and the inequality $\chi \leq \varepsilon^{p-2}$ we obtain

$$\|\chi\nabla\cdot\eta^2\nabla u\|_2 \leq \varepsilon^{p-2}(\lambda\|u\|_2 + \|f\|_2) + \|\chi|\eta\nabla u||b\eta|\,\|_2. \qquad (3.80)$$

Choosing $\varepsilon := \|f\|_\infty$ and combining Lemma 3.4.3 with (3.80) we infer that

$$\begin{aligned}\Big\|\,|\eta\nabla u|_\varepsilon^{p-1}|\eta\nabla u|\,\Big\|_2^2 &\leq C_p\Big\|f\Big\|_\infty^{2p-2}\Big((\lambda^2+1)\|u\|_2^2 + \|f\|_2^2\Big)\\ &\quad + C_p\Big\|f\Big\|_\infty^2\Big\|\chi|\eta\nabla u|(|b\eta| + |\beta\eta| + |\nabla\eta|)\,\Big\|_2^2.\end{aligned}$$

Next we apply the inequalities $\|f\|_p^p \geq \|f\|_2^2\|f\|_\infty^{p-2}$ for $p < 2$ and (3.66) to the last estimate and get

$$\begin{aligned}\Big\|\,|\eta\nabla u|_\varepsilon^{p-1}|\eta\nabla u|\,\Big\|_2^2 &\leq C_p\Big\|f\Big\|_\infty^2\Big\|\chi|\eta\nabla u|(|b\eta| + |\beta\eta| + |\nabla\eta|)\,\Big\|_2^2\\ + C_p\Big\|f\Big\|_\infty^p\Big\|f\Big\|_p^p\Big(\frac{1}{\lambda^2}+1\Big) &+ C_p\Big\|f\Big\|_\infty^2\Big\|\,|b\eta| + |\beta\eta|\,\Big\|_{2p}^{2p}\Big(\frac{1}{\lambda^p} + \frac{1}{\lambda^{p+2}}\Big). \qquad (3.81)\end{aligned}$$

Observe that $(\chi|\eta\nabla u|)^{p'} \leq |\eta\nabla u|_\varepsilon^{p-1}|\eta\nabla u|$ since $p < 2$. Hence, the Hölder inequality implies that

$$\|\chi|\eta\nabla u|g\|_2^2 \leq \|\,|\eta\nabla u|_\varepsilon^{p-1}|\eta\nabla u|\,\|_2^{\frac{2}{p'}}\|g\|_{2p} \quad \forall g \in L^{2p}(\Omega). \qquad (3.82)$$

In order to complete the proof we apply (3.82) and the Young inequality to (3.81). □

The next simple proposition provides an a priori estimate in $L^2(\Omega)$ for the gradient of the solution of (3.64).

Proposition 3.4.4. *Let* $1 < p \leq 3/2$. *Let* $\beta \in L^2_{\text{loc}}$. *Let* u *be the solution of* (3.64), *with* $b \in C^\infty(\overline{\Omega})$, $f \in C(\overline{\Omega})$ *and* $\lambda > 0$. *Then there exists a constant* C *such that*

$$\|\eta \nabla u\|_2^2 \leq C\|f\|_\infty^2 \Big(\lambda^{-1}\rho(\Omega) + \lambda^{-2}\|(b-\beta)\eta\|_2^2\Big), \tag{3.83}$$

where $\rho(\Omega) := \int_\Omega \rho(x)dx$.

Proof. Integrating by parts and using the equation we obtain

$$\|\eta \nabla u\|_2^2 = \langle u,\, f - \lambda u\rangle + \langle u,\, (b-\beta)\cdot \eta^2 \nabla u\rangle. \tag{3.84}$$

Estimating the right-hand side of (3.84) as follows

$$\begin{aligned} |\langle u,\, f - \lambda u\rangle| &\leq 2\lambda^{-1}\|f\|_\infty^2 \rho(\Omega)\,, \\ |\langle u,\, (b-\beta)\eta^2 \nabla u\rangle| &\leq \frac{1}{2}\lambda^{-2}\|f\|_\infty^2 \|(b-\beta)\eta\|_2^2 + \frac{1}{2}\|\eta\nabla u\|_2^2, \end{aligned}$$

one completes the proof. □

Chapter 4

L^p-Uniqueness for Infinite Dimensional Symmetric Dirichlet Operators with Variable Diffusion Coefficients

In this chapter we study infinite dimensional operators of the form

$$\hat{\mathcal{L}}u = -\sum_{k,j} a_{kj}\frac{\partial^2 u}{\partial x_j \partial x_k} - \sum_{k,j}\frac{\partial a_{kj}}{\partial x_k}\frac{\partial u}{\partial x_j} - \sum_{k,j}\beta_k^\mu a_{kj}\frac{\partial u}{\partial x_j},$$

where $u \in \mathcal{F}C_b^\infty$, i.e. the set of smooth finitely based functions on a locally convex vector space X. Here the entries of the symmetric positive definite matrix $a = (a_{kj})_{k,j\geq 1}$ satisfy certain regularity conditions and $\beta^\mu = (\beta_k^\mu)_{k\geq 1}$ stands for the logarithmic derivative of a probability measure μ on X. First we construct the generator $-\mathcal{L}_p$ of a C_0-semigroup on $L^p(X,\mu)$, $1 \leq p < \infty$, such that $\mathcal{L}_p \supset \hat{\mathcal{L}}$. The operator $\mathcal{L}_p$ is called the Dirichlet operator in $L^p(X,\mu)$. Then we reveal sufficient conditions on a and β^μ which ensure that the property of strong L^p-uniqueness holds for the operator $\mathcal{L}$, i.e. that $-\mathcal{L}_p$ is the only extension of $-\hat{\mathcal{L}}$ generating a C_0-semigroup on $L^p(X,\mu)$.

The chapter is organised as follows. Our main results are formulated in Theorems 4.1.2 and 4.1.3 in section 4.1 to which we also refer for the precise framework. In section 4.2 we provide the proofs of the uniqueness results, which are based on a priori estimates for the first order derivatives of solutions of parabolic equations with smooth coefficients. These estimates are derived in section 4.3, whereas certain auxiliary results are contained in section 4.5. An application, which could not be treated by previous results on the strong uniqueness, is given in section 4.4.

Most of the results, stated in this chapter, are contained in [50].

4.1 Framework and Main Results

Let X be a separable locally convex Hausdorff topological vector space such that its topological dual X^* contains a sequence $(l_n)_{n\in\mathbb{N}}$ of linearly independent functionals separating points. We assume that X is Souslin, hence l_n, $n \in \mathbb{N}$, generate the Borel σ-algebra $\mathcal{B}(X)$ of X (cf. [68]).

Given $N \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{\infty\}$, let $UC_b^m := UC_b^m(\mathbb{R}^N)$ stand for the class of m-times differentiable functions on $\mathbb{R}^N$, whose derivatives up to order m are bounded and uniformly continuous. Now let

$$\mathcal{F}C_b^{m,u}(\mathbb{R}^N) := \{f(l_1,\dots,l_N) : f \in UC_b^m(\mathbb{R}^N)\}.$$

From now on $\mathcal{F}C_b^{m,u} := \bigcup_N \mathcal{F}C_b^{m,u}(\mathbb{R}^N)$ and $\mathcal{F}C_b^\infty := \mathcal{F}C_b^{\infty,u}$.

Let μ be a probability measure on $\mathcal{B}(X)$. Suppose that $\operatorname{supp}\mu = X$. For $p \geq 1$ we set $L^p := \operatorname{Re} L^p(X, \mathcal{B}(X), \mu)$. Since $\mathcal{B}(X) = \sigma(l_n,\ n \in \mathbb{N})$, the set $\mathcal{F}C_b^\infty$ is dense in L^p for all $p \in [1, \infty)$. Throughout the paper we use the following notation: $\|\cdot\|_p$ is the norm in L^p, $\langle\cdot,\cdot\rangle$ is the inner product in L^2, and $\langle f\rangle := \int_X f d\mu$.

Let $(e_k)_{k\in\mathbb{N}} \subset X$ be the unique sequence of linearly independent vectors such that $l_m(e_k) = \delta_{mk}$, $m, k \in \mathbb{N}$. We assume that the measure μ is differentiable along every e_k in the sense that there exist measurable functions $(\beta_k^\mu)_{k\in\mathbb{N}}$ in L^2, satisfying

$$\left\langle \frac{\partial v}{\partial e_k} \right\rangle = -\langle v,\ \beta_k^\mu\rangle,\quad v \in \mathcal{F}C_b^{1,u},\ k \in \mathbb{N}.$$

For every $k \in \mathbb{N}$ the distribution β_k^μ is called directional logarithmic derivative of μ along e_k. Further on we treat $(e_k)_{k\in\mathbb{N}}$ as the canonical basis in the space $\mathbb{R}^{\mathbb{N}}$ of all real sequences. Hence, we identify the linear span of $(e_k)_{k\in\mathbb{N}}$ with the space $\mathbb{R}^{\text{fin}}$ of all finite sequences. The space $\mathbb{R}^{\text{fin}}$ can be considered as the tangent space T_xX to X for all $x \in X$ in the sense that we shall take derivatives only along the elements of $\mathbb{R}^{\text{fin}}$. We introduce the spaces $(H_0, (\cdot,\cdot)_0) = l^2$, $(H_+, (\cdot,\cdot)_+) = l^2_{\gamma_k}$ and $(H_-, (\cdot,\cdot)_-) = l^2_{\gamma_k^{-1}}$ for a sequence $(\gamma_k)_{k\in\mathbb{N}}$ in $(0, +\infty)$ (where $l^2_{\gamma_k} := \{h \in \mathbb{R}^{\mathbb{N}} : \sum_k h_k^2\gamma_k^2 < \infty\}$ and $l^2_{\gamma_k^{-1}}$ is defined in the same way). It is obvious that H_+ and H_- are mutually dual w.r.t. the $(\cdot,\cdot)_0$-duality pairing.

For $N \in \mathbb{N}$ we define the projection $P_N : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\text{fin}}$:

$$P_N h := \sum_{k=1}^{N} h_k e_k.$$

Here and below we denote the linear span of $(e_k)_{k=1}^N$ by $\mathbb{R}^N$. This notation is consistent with the definition of $\mathcal{F}C_b^{m,u}(\mathbb{R}^N)$ since for $x \in X$ we have

$$P_N x := \sum_{k=1}^{N} l_k(x) e_k.$$

For $u \in \mathcal{F}C_b^{1,u}$ such that $u(\cdot) = f(P_N \cdot)$ let ∇u stand for the Frechet derivative of u, i.e.

$$\nabla u(x) := \sum_{k=1}^{N} \frac{\partial f}{\partial e_k}(P_N x) e_k \in \mathbb{R}^N.$$

Furthermore, if $u \in \mathcal{F}C_b^{2,u}$, then by $D^2 u$ we denote the second derivative of u:

$$D^2 u(x) := \sum_{k,m=1}^{N} \frac{\partial^2 f}{\partial e_k \partial e_m}(P_N x) e_k \otimes e_m, \; x \in X.$$

Further on we use the same notation as in Chapter 3: $\nabla_k := \frac{\partial}{\partial e_k}$, $\nabla_{kj}^2 := \frac{\partial^2}{\partial e_k \partial e_j}$.

Let $\{a_{kj},\, k, j \in \mathbb{N}\}$ be a family of cylindrical functions on X. The following conditions **(A0)** - **(A4)** on (a_{kj}) are assumed to hold throughout the chapter.

(A0) For every $N \in \mathbb{N}$ the matrix $(a_{kj}(x))_{k,j=1}^N$ is symmetric and uniformly elliptic. For every $i \in \mathbb{N}$ there exists $\varepsilon_i \in (0, \infty)$, such that for μ-a.e. $x \in X$,

$$\sum_{k,j=1}^{\infty} a_{kj}(x) h_k h_j \geq \varepsilon_i h_i^2, \quad \text{for all } \; h = (h_k)_{k \in \mathbb{N}} \in \mathbb{R}^{\text{fin}},$$

and the completion $H_a(x)$ of $\mathbb{R}^{\text{fin}}$ with respect to the norm $\|\cdot\|_a := (\cdot,\cdot)_a^{1/2}$, where

$$(h, g)_a(x) := \sum_{k,j=1}^{\infty} a_{kj}(x)\, h_k\, g_j, \qquad g, h \in \mathbb{R}^{\text{fin}},$$

embeds one-to-one and continuously into $\mathbb{R}^{\mathbb{N}}$ (the latter being equipped with the product topology).

Note that assumption **(A0)** is fulfilled if the infinite matrix$(a_{kj}(x))_{k,j=1}^{\infty}$ is block diagonal and each block is uniformly elliptic. By HS and $HS(a)$ we denote the spaces of Hilbert-Schmidt operators over H_0 and H_a respectively.

(A1) For every $n \in \mathbb{N}$

$$a_{kj} \in \mathcal{F}C_b^{1,u}(\mathbb{R}^{K_n}),\ k,j = 1,\dots,K_n,$$

for a sequence $(K_n)_{n\in\mathbb{N}} \subset \mathbb{N},\ K_n \nearrow \infty$.

(A2) For every $k \in \mathbb{N}$

$$\bar{c}_k := \sup_{x\in X, j\in\mathbb{N}} a_{kj}^2(x) < \infty.$$

For every $k \in \mathbb{N}$ we assume that β_k^μ can be decomposed as $\beta_k^\mu = \xi_k^\mu + \eta_k^\mu$, with ξ_k^μ and η_k^μ Borel measurable and satisfying the following conditions.

(A3) The series $\eta^\mu(x) := \sum_{k\in\mathbb{N}} \eta_k^\mu(x) e_k$ converges in $H_a(x)$ for a.e. $x \in X$ and $|\eta^\mu - P_N\eta^\mu|_a \to 0$ in L^2 as $N \to \infty$.

Note that, for all $k \leq N < d$, the Cauchy inequality gives

$$\left|\sum_{j=N}^{d} a_{kj}\eta_j^\mu\right| \leq \left(\sum_{l=1}^{d} \alpha_{kl}^2\right)^{\frac{1}{2}} \left(\sum_{l=1}^{d}\left(\sum_{j=N}^{d} \alpha_{jl}\eta_j^\mu\right)^2\right)^{\frac{1}{2}} = \sqrt{a_{kk}}\,|(P_d - P_N)\eta^\mu|_a,$$

where $(\alpha_{jl})_{j,l=1}^d$ is the square root of the matrix $(a_{jl})_{j,l=1}^d$. Therefore **(A3)** implies that the series $\sum_{j\geq 1} a_{kj}\eta_j^\mu$ converges in L^2 for all $k \in \mathbb{N}$.

(A4) The series $\xi_{a;k}^\mu := \sum_{j\in\mathbb{N}} (\nabla_j a_{kj} + a_{kj}\xi_j^\mu)$ is convergent in L^2 for all $k \in \mathbb{N}$.

The latter enables us to introduce $\xi_a^\mu(x) := \sum_{k\geq 1} \xi_{a;k}^\mu(x) e_k \in \mathbb{R}^{\mathbb{N}}$. The vector ξ_a^μ is referred to as the "large" part of the collection $(\beta_k^\mu)_{k\in\mathbb{N}}$ of the directional logarithmic derivatives of μ, since it is not a section of the "co-tangent bundle" $(H_a(x))_{x\in X}$ (see [19, Appendix D] for precise definition).

For $d \in \mathbb{N}$ and $x \in X$ we introduce the quantities $v_d(x)$ and $\nu_d(x)$:

$$v_d^2(x) := \sup_{|h|_a \leq 1} \sum_{i,j,k,l,m,n=1}^{d} a_{kl}(x) a^{mj}(x) (\nabla_k a_{ij}(x) h_i)(\nabla_l a_{mn}(x) h_n), \tag{4.1}$$

and

$$\nu_d^2(x) := \sup_{|h|_0 \leq 1} \sum_{i,j,k,l,m=1}^{d} (\gamma_l^{-2} a^{ij}(x) (\nabla_k a_{il}(x) \gamma_k h_k)(\nabla_m a_{jl}(x) \gamma_m h_m), \tag{4.2}$$

where $h = (h_n)_{n\in\mathbb{N}}$ and $(a^{ij})_{i,j=1}^d$ is the matrix inverse to $(a_{ij})_{i,j=1}^d$ (which exists due to **(A1)**). Note that if $a_{kj}(x) = \delta_{kj}\sigma_k(x)$, one has

$$v_d^2(x) = \sup_m \sum_{l=1}^d \sigma_l(x)\left(\frac{\nabla_l \sigma_m(x)}{\sigma_m(x)}\right)^2.$$

If one assumes, in addition, that $\sigma_k(x) = \sigma_k(x_k)$, then

$$v_d(x) = \sup_k \sigma_k^{-\frac{1}{2}}(x_k)|\sigma_k'(x_k)|$$

and

$$\nu_d(x) = \sup_k \frac{|\sigma_k'(x_k)|}{\sigma_k(x_k)}.$$

We define an operator $\hat{\mathcal{L}}$ with the domain $\mathcal{D}(\hat{\mathcal{L}}) := \mathcal{F}C_b^\infty$ in L^2 by

$$\hat{\mathcal{L}}v = \sum_{k,j\geq 1}(a_{kj}\nabla_{kj}^2 v + a_{kj}\eta_j^\mu \nabla_k v) + \sum_{k\geq 1}\xi_{a;k}^\mu \nabla_k v.$$

We observe that v is cylindric, so it follows from **(A1)**, **(A3)** and **(A4)** that $\hat{\mathcal{L}}v \in L^2$ for all $v \in \mathcal{F}C_b^\infty$. Hence, the operator $\hat{\mathcal{L}}$ is densely defined in L^2. We also note that the following equality holds:

$$-\langle \hat{\mathcal{L}}u,\, v\rangle \;=\; \sum_{k,j\geq 1}\langle a_{kj}\nabla_k u \nabla_j v\rangle,\ u,\, v \in \mathcal{F}C_b^\infty. \tag{4.3}$$

Indeed, for $u,\, v \in \mathcal{F}C_b^{2,u}$ we have

$$-\int_X v\Big[\sum_{k,j=1}^\infty a_{kj}\nabla_{kj}^2 u + \sum_{k=1}^\infty \nabla_k u\Big(\sum_{j=1}^\infty(\nabla_j a_{kj} + a_{kj}\xi_j^\mu) + \sum_{j=1}^\infty a_{kj}\eta_j^\mu\Big)\Big]d\mu.$$
$$= -\sum_{k,j=1}^\infty \int_X v\,(\nabla_j + \beta_j^\mu)(a_{jk}\nabla_k u)d\mu,$$

since the sum in k is finite and the series in j converges in L^2 due to **(A3)** and **(A4)**. Hence, (4.3) follows from the integration by parts formula. Therefore, $\hat{\mathcal{L}}$ is a symmetric operator and the form

$$\mathcal{E}(u,\, v) \;=\; \sum_{k,j\geq 1}\langle a_{kj}\nabla_k u\nabla_j v\rangle,\ u,\, v \in \mathcal{F}C_b^\infty$$

is a closable symmetric form on L^2 whose closure $(\mathcal{E},\, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form. We will not distinguish between $\mathcal{E}$ and its closure unless it leads to confusions.

As described in subsection 2.2.3, the form $\mathcal{E}$ gives rise to a family of consistent sub-Markovian C_0-semigroups of contractions $\exp(-\mathcal{L}_p t)$ on L^p, $1 \leq p < \infty$. By construction $\mathcal{L}_2 \supset \hat{\mathcal{L}}$ (note that $\mathcal{L}_2 \equiv \mathcal{L}$ is the Friedrichs extension of $\mathcal{L}$). Furthermore, the following simple statement holds.

Lemma 4.1.1. *Let $s = 2 \vee p$. Then $\mathcal{L}_p \supset \hat{\mathcal{L}}$ provided $|\eta^\mu|_a$, $\xi^\mu_{a;k} \in L^s$ for all $k \in \mathbb{N}$.*

Proof. Our assumptions imply that $\hat{\mathcal{L}}v \in L^2 \cap L^p$ for all $v \in \mathcal{F}C_b^\infty$. Therefore,

$$-\hat{\mathcal{L}}v = L^p\text{-}\lim_{t\to 0} \frac{1}{t} \int_0^t e^{-\mathcal{L}_p s} \mathcal{L}v ds.$$

On the other hand,

$$\int_0^t e^{-\mathcal{L}_p s}\mathcal{L}vds = \int_0^t e^{-\mathcal{L}_2 s}\mathcal{L}vds = -\int_0^t e^{-\mathcal{L}_2 s}\mathcal{L}_2 vds = v - e^{-\mathcal{L}_2 t}v = v - e^{-\mathcal{L}_p t}v.$$

Thus, $\hat{\mathcal{L}}v = L^p\text{-}\lim_{t\to 0}\frac{1}{t}(v - e^{-\mathcal{L}_p t}v)$ and $\mathcal{L}_p \supset \hat{\mathcal{L}}$. □

Now we are ready to formulate the uniqueness results. Recall that conditions **(A0) - (A4)** are still in force.

Theorem 4.1.2. *Let $p \geq 1$, (K_n) be as in* **(A1)**. *Set $s := p$ if $\eta^\mu \equiv 0$, and $s := max(2,p)$ otherwise. Let $\xi^\mu_{a;k} \in L^s$ for all $k \in \mathbb{N}$ and $\sup_n \| \nu_n \|_\infty < \infty$.*

Assume that

(i) there exists a sequence $\xi_j^n \in \mathcal{F}C_b^{1,u}(\mathbb{R}^{K_n})$, $j = 1, \dots, K_n$, $n \in \mathbb{N}$, such that

(a) $|\xi^n - P_{K_n}\xi^\mu_a|_- \to 0$ in L^s as $n \to \infty$;

(b) there exists a constant $c_+ \in \mathbb{R}$ independent of n such that for all x, $y \in \mathbb{R}^{K_n}$ we have

$$\sum_{j,l=1}^{K_n} \gamma_l^2 (\nabla_l \xi_j^n)(x) y_j y_l \leq c_+ \sum_{k=1}^{K_n} \gamma_k^2 y_k^2;$$

(ii) either $\eta^\mu = 0$

or

(a) $\sup_d \| v_d \|_{2p} < \infty$, $|\eta^\mu - P_N \eta^\mu|_a \to 0$ in L^{2p} as $N \to \infty$;

(b) there exist a sequence $(\hat{\xi}_j^m)_{m,j\in\mathbb{N}} \subset \mathcal{F}C_b^{1,u}$ and numbers $\varepsilon_0 \in [0,1)$ and $c(\varepsilon_0) \in \mathbb{R}$ such that $\hat{\xi}_j^m \to \xi_j^\mu$ as $m \to \infty$ weakly in L^2 for every $j \in \mathbb{N}$, and for all $n \in \mathbb{N}$ and $w_j \in \mathcal{F}C_b^{1,u}(\mathbb{R}^n)$, $j = 1, \dots, n$, the following inequality holds

$$\liminf_{m,d\to\infty} \sum_{i,k=1}^{d} \sum_{j,l=1}^{n} \langle (\nabla_k \hat{\xi}_j^m) a_{jl} w_l,\, a_{ki} w_i \rangle$$

$$\leq \varepsilon_0 \sum_{i,j,k,l=1}^{n} \langle a_{kj} \nabla_i w_j,\, a_{il} \nabla_k w_l \rangle + c(\varepsilon_0) \sum_{j,k=1}^{n} \langle a_{kj} w_k,\, w_j \rangle;$$

(c) $p \in \left(3 - \dfrac{3}{1+\sqrt{1+3\varepsilon_0}}, \dfrac{2}{\varepsilon_0}\right)$.

Then the operator $\hat{\mathcal{L}} \restriction_{\mathcal{F}C_b^\infty}$ has a unique extension which generates a C_0-semigroup on L^p.

Next we formulate the uniqueness result in L^1.

Theorem 4.1.3. *Let (K_n) be as in* **(A1)**. *Let $\xi^\mu_{a;k} \in L^1$ for all $k \in \mathbb{N}$ and* $\sup_n \| \nu_n \|_\infty < \infty$.

Assume that there exists a sequence $\xi^n_j \in \mathcal{F}C_b^{1,u}(\mathbb{R}^{K_n})$, $j = 1, \dots, K_n$, $n \in \mathbb{N}$, such that

(a) $|\xi^n - P_{K_n}\xi^\mu_a|_- \to 0$ *in L^1 as $n \to \infty$;*

(b) *there exists a constant $c_+ \in \mathbb{R}$ independent of n such that for all x, $y \in \mathbb{R}^{K_n}$ we have*
$$\sum_{j,l=1}^{K_n} \gamma_l^2 (\nabla_l \xi^n_j)(x) y_j y_l \le c_+ \sum_{k=1}^{K_n} \gamma_k^2 y_k^2.$$

Then the operator $\hat{\mathcal{L}} \restriction_{\mathcal{F}C_b^\infty}$ has a unique extension which generates a C_0-semigroup on L^1.

Remark. *The uniqueness results in [48] (Theorems 1 and 3) can be obtained as particular cases of Theorem 4.1.2 if one puts $a_{jk}(x) \equiv \delta_{jk}$. (Note the difference in the interval in the L^p-scale, which was incorrectly stated in [48], Theorem 3.) Theorem 4.1.3 generalises Theorem 2 from [48]. In [49] the special case $\xi^\mu_k = 0$, $k \in \mathbb{N}$, was studied and strong L^p-uniqueness of the extension of $\hat{\mathcal{L}}$ has been proved under weaker assumptions on the coefficients a_{jk}, namely, their derivatives need not be either continuous or bounded. If we confine ourselves to this situation then we can employ estimates* (4.7) *and* (4.8) *(see Proposition 4.2.3 below and note that in this case $\varepsilon_0 = 0$) and prove the uniqueness under the same assumptions as in [49].*

4.2 Proof of Uniqueness

Our strategy to prove the uniqueness result is as follows. We take an arbitrary extension $\mathcal{B} \supset -\hat{\mathcal{L}} \restriction_{\mathcal{F}C_b^\infty}$, which generates a C_0-semigroup on L^p. Then we take sequences $(\xi_j^m), (\eta_j^m) \subset \mathcal{F}C_b^{1,u}$ and investigate the corresponding family of Cauchy problems:

$$\begin{cases} \partial_t u^{(m)} & = \sum_{k,j=1}^{K_m} (a_{jk} \nabla_{kj}^2 u^{(m)} + a_{kj} \eta_j^m \nabla_k u^{(m)}) + \sum_{k \geq 1} \xi_k^m \nabla_k u^{(m)} \\ u^{(m)}(0) & = f, \end{cases}$$

with an arbitrary $0 \neq f \in \mathcal{F}C_b^\infty$ and (K_m) as in **(A1)**. Then we show that $u^{(m)}(t) \to e^{-\mathcal{B}t} f$ strongly in L^p provided ξ_j^m, η_j^m approximate $\xi_{a;j}^\mu$, η_j^μ, $j \in \mathbb{N}$ in a proper way. This will prove strong uniqueness for the generator. The core of the proof is a priori estimates for the gradient ∇u of the solution u to the following Cauchy problem over $\mathbb{R}^K$

$$\begin{cases} \partial_t u & = \mathcal{L}_{\xi,\eta} u := \sum_{k,j=1}^{K} (a_{kj} \nabla_{kj}^2 u + a_{kj} \eta_j \nabla_k u) + \sum_{k=1}^{K} \xi_k \nabla_k u, \quad t > 0, \\ u(0) & = f, \end{cases} \tag{4.4}$$

with a uniformly elliptic matrix $(a_{jk})_{j,k=1}^K$, a_{jk}, $\xi_j \in UC_b^1(\mathbb{R}^K)$, $j,k = 1,\dots,K$, $N < K$, $\eta_j \in UC_b^1(\mathbb{R}^N)$, $j = 1,\dots,N$, $\eta_j \equiv 0$, $j = N+1,\dots,K$, $f \in C_b^\infty(\mathbb{R}^K)$.

In order to obtain the required estimates we need a result from [57]. To formulate it we introduce the notion of classical solution to the abstract Cauchy problem (see [57, Definition 4.1.1, (iii)]). We consider the problem

$$v'(t) = \mathcal{A} v(t) + f(t), \; t > 0; \quad v(0) = v_0, \tag{4.5}$$

where $\mathcal{A}$ is a linear sectorial operator in a Banach space Y. The function f is assumed to be continuous in $(0,\infty)$. A function $v : [0,\infty) \to Y$ is said to be a *classical* solution to problem (4.5) in $[0,\infty)$, if for every $T > 0$ the function $v \in C^1((0,T],Y) \cap C((0,T],\mathcal{D}(\mathcal{A})) \cap C([0,T],Y)$, $v'(t) = \mathcal{A}v(t) + f(t)$ for all $0 < t \leq T$, and $v(0) = v_0$.

Proposition 4.2.1. *([57, Propositions 3.1.9, 3.1.17, 3.1.18].) Set*

$$\mathcal{D} = \{u \in \bigcap_{p \geq 1} W_{\mathrm{loc}}^{2,p}(\mathbb{R}^K) : u, \; \mathcal{L}_{\xi,\eta} u \in C_b(\mathbb{R}^K)\}.$$

Then

(i) $\mathcal{L}_{\xi,\eta} \restriction_{\mathcal{D}}$ generates a positive analytic semigroup $U(t)$ on C_b, which is continuous at zero on elements from $\overline{\mathcal{D}} = UC_b(\mathbb{R}^K)$. In particular, problem (4.4) has a unique classical solution $u \in C_b$ (in the sense of ;

(ii) the functions $t \mapsto u(t)$ *and* $t \mapsto \mathcal{L}_{\xi,\eta}u(t)$ *are analytic* $(0,\infty) \to C_b^1(\mathbb{R}^K)$ *and* $u(t) \to f$, $\mathcal{L}_{\xi,\eta}u(t) \to \mathcal{L}_{\xi,\eta}f$ *in* $C_b^1(\mathbb{R}^K)$ *as* $t \to 0$;

(iii) For all $t \geq 0$ *we have* $u(t) \in \bigcap_{p\geq 1} W^{3,p}_{\mathrm{loc}} \bigcap UC_b^2$.

By the maximum principle we have $\|u\|_\infty \leq \|f\|_\infty$.

The a priori estimates are given in the following three propositions.

Proposition 4.2.2. *Let* u *be the solution to* (4.4). *Assume that there exists a constant* c_+ *independent of* x *such that for all* x, $y \in \mathbb{R}^K$ *the inequalities*

$$\sum_{j,l=1}^K \gamma_l^2 (\nabla_l \xi_j)(x) y_j y_l \leq c_+ \sum_{k=1}^K \gamma_k^2 y_k^2,$$

$$\nu_K(x) \leq c_+$$

hold (ν_K *is as in (4.2)).*

Then

$$\| \, |\nabla u|_+ \|_\infty \leq \exp(C_+ t) \| \, |\nabla f|_+ \|_\infty,$$

with $C_+ = c_+ + \frac{1}{4}c_+^2 + \left(\sum_{k=1}^N \bar{c}_k\right)^{\frac{1}{2}} \| \, |\nabla \eta|_{HS}\|_\infty + c_+ \| \, |\eta|_a \|_\infty$ *and* $\eta = (\eta_1, \ldots, \eta_K)$ *(recall that* $|h|_+^2 = \sum_{k\geq 1} \gamma_k^2 h_k^2$, $h \in H_+$*).*

Proposition 4.2.3. *Let* u *be the solution to* (4.4). *For* $3 - \frac{3}{1+\sqrt{1+3\varepsilon_0}} < p < \frac{2}{\varepsilon_0}$ *set* $s = max(p, 2)$. *Let* $\sup_d \|v_d\|_{2p} < \infty$, $|\eta^\mu - P_d \eta^\mu|_a \to 0$ *in* L^{2p} *as* $d \to \infty$ *and* $\xi^\mu_{a;k} \in L^s$, $k = 1, \ldots, K$. *Set* $G_p := \| \, |\eta|_a + |\eta^\mu|_a \|_{2p}^{2p} + \sup_d \|v_d\|_{2p}^{2p}$. *Let* C_+ *be as in Proposition 4.2.2. Then there exists a constant* $C_{\varepsilon_0,p} > 0$, *depending only on* p *and* ε_0, *such that*

$$\int_0^t \| \, |\nabla u|_a \|_{2p}^{2p}(\tau) d\tau \leq C_{\varepsilon_0,p} \Bigg[t \|f\|_\infty^{2p} (G_p + 1) + \|f\|_\infty^2 \| \, |\nabla f|_a \|_{2p-2}^{2p-2}$$
$$+ \frac{e^{sC_+t} - 1}{sC_+} \|f\|_\infty^p \| \, |\nabla f|_+ \|_\infty^p \| \, |P_K \xi_a^\mu - \xi|_- \|_p^p \Bigg]. \tag{4.6}$$

(Recall that $|\nabla u|_a^2 = \sum_{k,j=1}^K a_{kj} \nabla_k u \nabla_j u$.*)*

Furthermore, for $p = 2$,

$$\int_0^t \| \, |AD^2 u|_{HS(a)} \|_2^2 (\tau) d\tau \leq C_{\varepsilon_0} \Bigg[\|f\|_\infty^2 t (G_2 + 1) + \| \, |\nabla f|_a \|_2^2$$
$$+ \frac{e^{2C_+t} - 1}{2C_+} \| \, |\nabla f|_+ \|_\infty^2 \| \, |P_K \xi_a^\mu - \xi|_- \|_2^2 \Bigg]. \tag{4.7}$$

and, for $3 - \frac{3}{1+\sqrt{1+3\varepsilon_0}} < p < 2$,

$$\int_0^t \| \, |AD^2u|_{HS(a)} \|_p^p(\tau) d\tau \le C_{p,\varepsilon_0} \Big[t\|f\|_\infty^p (G_p + 1) + \|f\|_\infty^{2-p} \| \, |\nabla f|_a \|_{2p-2}^{2p-2} + \|f\|_\infty^{p-2} \frac{e^{2C_+ t} - 1}{2C_+} \| \, |\nabla f|_+ \|_\infty^2 \| \, |P_K \xi_a^\mu - \xi|_- \|_2^2 \Big], \tag{4.8}$$

where $|AD^2u|_{HS(a)}^2 = \sum_{i,j,k,m=1}^K a_{ij}a_{km}\nabla_{jk}^2 u \nabla_{mi}^2 u$.

Proposition 4.2.4. *Let u be the solution to (4.4). Let C_+ be as in Proposition 4.2.2. We assume that $\xi_{a;k}^\mu \in L^1$, $k = 1, \dots, K$. Then there exists a constant $C > 0$ such that*

$$\int_0^t \| \, |\nabla u|_a \|_2^2(\tau) d\tau \le C \Big[t\|f\|_\infty^2 \| \, |\eta - P_K \eta^\mu|_a \|_2^2 + \|f\|_\infty^2 + \frac{e^{C_+ t} - 1}{C_+} \|f\|_\infty \| \, |\nabla f|_+ \|_\infty \| \, |P_K \xi_a^\mu - \xi|_- \|_1 \Big].$$

We postpone the proof of Propositions 4.2.2–4.2.4 till the next section.

Proof of Theorem 4.1.2. Let $f \in \mathcal{F}C_b^\infty$. For $N \ge 1$ let $\eta_j^N \in \mathcal{F}C_b^\infty(\mathbb{R}^N)$, $j = 1, \dots, N$, satisfy $\| \, |\eta^\mu - \eta^N|_a \|_{2p} \le 1/N$, with $\eta^N := \sum_j \eta_j^N e_j$.

Let (ξ_k^n), $k = 1, \dots, K_n$, $n \in \mathbb{N}$, be the sequence satisfying condition (i) of the theorem. We choose n to be such that $K_n \ge N$.

By $u^{(Nn)}$ we denote the solution to the Cauchy problem on $\mathbb{R}^{K_n}$

$$\begin{cases} \partial_t u^{(Nn)} &= -\mathcal{L}_{\xi^n, \eta^N} u^{(Nn)} =: -\mathcal{L}_{Nn} u^{(Nn)}, \\ u^{(Nn)}(0) &= f \end{cases} \tag{4.9}$$

Let $\mathcal{B}$ with $\mathcal{D}(\mathcal{B})$ stand for an arbitrary extension of $\hat{\mathcal{L}} \restriction_{\mathcal{F}C_b^\infty}$, such that $-\mathcal{B}$ generates a C_0-semigroup on L^p. It is easy to show that $\mathcal{D}(\mathcal{B}) \supset \mathcal{F}C_b^{2,u}$ and

$$-\mathcal{B}u = \sum_{k,j \ge 1} (a_{kj}\nabla_{kj}^2 u + a_{kj}\eta_j^\mu \nabla_k u) + \sum_{k \ge 1} \xi_{a;k}^\mu \nabla_k u, \quad u \in \mathcal{F}C_b^{2,u}.$$

By Proposition 4.2.1, (iii) the function $u^{(Nn)}(t) \in \mathcal{D} \subset \mathcal{D}(\mathcal{B})$ for all $t \ge 0$. Therefore the function $s \mapsto e^{-\mathcal{B}(t-s)} u^{(Nn)}(s)$ is a continuously differentiable map $[0, t] \to L^p$ (see Remark 2.2.2, (ii)). Making use of Proposition 4.2.1, (ii) we obtain the Duhamel formula:

$$u^{(Nn)}(t) - e^{-t\mathcal{B}} f = e^{-(t-\tau)\mathcal{B}} u^{(Nn)}(\tau)\Big|_{\tau=0}^{\tau=t} = \int_0^t e^{-(t-\tau)\mathcal{B}} (\mathcal{B} - \mathcal{L}_{Nn}) u^{(Nn)}(\tau) d\tau.$$

By Remark 2.2.2, (iii) there exist numbers $M \geq 1$ and $\gamma \geq 0$, such that for every $g \in L^2 \cap L^p$ we have $\|e^{-t\mathcal{B}}g\|_p \leq Me^{t\gamma}\|g\|_p$. Thus we get

$$\|e^{-t\mathcal{B}}f - u^{(Nn)}(t)\|_p \leq Me^{t\gamma}\Big(\| \, |\xi^n - P_{K_n}\xi^\mu_a|^2_- \|_s \int_0^t \| \, |\nabla u^{(Nn)}|_+\|_\infty d\tau$$

$$+ \| \, |\eta^\mu - \eta^N|_a\|_{2p} \int_0^t \| \, |\nabla u^{(Nn)}|_a\|_{2p} d\tau\Big). \tag{4.10}$$

In order to complete the proof of the theorem we need to show that

$$\lim_{N\to\infty} \limsup_{n\to\infty} \|e^{-t\mathcal{B}}f - u^{(Nn)}(t)\|_p = 0.$$

If $\eta^\mu_k = 0$ for all k, then one can take $\eta^N_k = 0$ and the result follows from Proposition 4.2.2 since $\| \, |\nabla u^{(Nn)}|_+\|_\infty(t) \leq e^{(c_+ + \frac{c_+^2}{4})t}\| \, |\nabla f|_+\|_\infty$.

In case $\eta^\mu_k \neq 0$ we employ Propositions 4.2.2 and 4.2.3 with the constant

$$C_+ = C_+(N) = c_+ + \frac{1}{4}c_+^2 + \big(\sum_{k=1}^N \bar{c}_k\big)\| \, |\nabla\eta^N|_{HS}\|_\infty + c_+\| \, |\eta^N|_a\|_\infty$$

to estimate $\| \, |\nabla u^{(Nn)}|_+\|_\infty$ and $\int_0^t \| \, |\nabla u^{(Nn)}|_a\|_{2p} ds$. Then we get

$$\limsup_{n\to\infty} \|e^{-t\mathcal{B}}f - u^{(Nn)}(t)\|_p \leq C\| \, |\eta^\mu - \eta^N|_a\|_{2p}\Big(\|f\|_\infty t G_p^{\frac{1}{2p}}$$

$$+ t^{\frac{2p-1}{2p}}\big(\|f\|_\infty \| \, |\nabla f|_a\|^{p-1}_{2p-2}\big)^{\frac{1}{p}}\Big).$$

Taking the limit as $N \to \infty$ we complete the proof. $\square$

Proof of Theorem 4.1.3. The proof is similar to that of Theorem 4.1.2.

Let $f \in \mathcal{F}C_b^\infty$. For $N \geq 1$ let $\eta_j^N \in \mathcal{F}C_b^\infty(\mathbb{R}^N)$, $j = 1, \ldots, N$, satisfy the estimate $\| \, |\eta^\mu - \eta^N|_a\|_2 \leq 1/N$, with $\eta^N := \sum_j \eta_j^N e_j$.

Let (ξ_k^n), $k = 1, \ldots, K_n$, be the sequence satisfying condition (i) of the theorem. We choose n in such a way that $K_n \geq N$.

By $u^{(Nn)}$ we denote the solution to problem (4.9) on $\mathbb{R}^{K_n}$, with ξ^n and η^N as above.

Let $\mathcal{B}$ with $\mathcal{D}(\mathcal{B})$ stand for an arbitrary extension of $\mathcal{L} \restriction_{\mathcal{F}C_b^\infty}$, such that $-\mathcal{B}$ generates a C_0-semigroup on L^1. It is easy to show that $\mathcal{D}(\mathcal{B}) \supset \mathcal{F}C_b^{2,u}$ and

$$-\mathcal{B}u = \sum_{k,j\geq 1}(a_{kj}\nabla^2_{kj}u + a_{kj}\eta^\mu_j\nabla_k u) + \sum_{k\geq 1}\xi^\mu_{a;k}\nabla_k u, \quad u \in \mathcal{F}C_b^{2,u}.$$

Repeating the corresponding argument from the proof of Theorem 4.1.2 we derive the Duhamel formula:

$$u^{(Nn)}(t) - e^{-t\mathcal{B}}f = e^{-(t-\tau)\mathcal{B}}u^{(Nn)}(\tau)|_{\tau=0}^{\tau=t} = \int_0^t e^{-(t-\tau)\mathcal{B}}(\mathcal{B} - \mathcal{L}_{Nn})u^{(Nn)}(\tau)d\tau.$$

Now we have

$$\|e^{-t\mathcal{B}}f - u^{(Nn)}(t)\|_1 \leq Me^{t\gamma}\Big[\| |\xi^n - P_{K_n}\xi_a^\mu|_-^2\|_1 \int_0^t \| |\nabla u^{(Nn)}|_+\|_\infty d\tau$$

$$+ \| |\eta^\mu - \eta^N|_a\|_2 \int_0^t \| |\nabla u^{(Nn)}|_a\|_2 d\tau\Big].$$

In order to complete the proof of the theorem we need to show that

$$\lim_{N\to\infty} \limsup_{n\to\infty} \|e^{-t\mathcal{B}}f - u^{(Nn)}(t)\|_1 = 0.$$

The rest of the proof is the same as that of Theorem 4.1.3. One has to apply Proposition 4.2.4 in place of Proposition 4.2.3 in order to estimate

$$\int_0^t \| |\nabla u^{(Nn)}|_a\|_2 d\tau.$$

□

4.3 Proof of A Priori Estimates

Throughout this section $(a_{kj})_{k,j=1}^K$, a_{kj}, $\xi_k \in \mathcal{F}C_b^{1,u}(\mathbb{R}^K)$, $k,j = 1,\ldots,K$, $\eta_i \in \mathcal{F}C_b^{1,u}(\mathbb{R}^N), i = 1,\ldots,N$ for some $N < K$, and $\eta_i \equiv 0$, $i = N+1,\ldots,K$, $f \in \mathcal{F}C_b^\infty(\mathbb{R}^K)$, $f \neq 0$; $u(t) \in \mathcal{F}C_b^{2,u}(\mathbb{R}^K)$, $t \geq 0$, is the solution to the Cauchy problem (4.4). Unless otherwise indicated, all the sums are from 1 to K. We also assume that the measure μ satisfies the conditions of Theorem 4.1.2.

We begin with formulating an auxiliary result from [57].

Proposition 4.3.1. *([57, Proposition 4.1.2].) Let Y be a Banach space. Let $f \in L^1((0,T),Y) \cap C((0,T],Y)$ and $u_0 \in \overline{\mathcal{D}(\mathcal{A})}$. If u is a classical solution to* (4.5), *then*

$$u(t) = \exp(t\mathcal{A})u_0 + \int_0^t \exp\left((t-s)\mathcal{A}\right)f(s)ds, \quad 0 \leq t \leq T.$$

We are now heading towards establishing the estimates for the derivatives of the solution to (4.4).

Proof of Proposition 4.2.2. Let us differentiate equation (4.4) in the direction e_k (observe that u is three times differentiable by Proposition 3.1, (iii)), then multiply by $\gamma_k^2 \nabla_k u$ and sum up from 1 to K. We get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sum_k \gamma_k^2(\nabla_k u)^2 &= \sum_{i,j,k}(a_{ij}(\nabla^3_{kij}u)\gamma_k^2\nabla_k u + \eta_i a_{ij}\nabla^2_{kj}u\gamma_k^2\nabla_k u) + \sum_{i,k}\xi_i\nabla^2_{ik}u\gamma_k^2\nabla_k u \\ &+ \sum_{i,j,k}((\nabla_k a_{ij})\nabla^2_{ij}u\gamma_k^2\nabla_k u + (\nabla_k\eta_i)a_{ij}\nabla_j u\gamma_k^2\nabla_k u) \\ &+ \sum_{i,j,k}\eta_i(\nabla_k a_{ij})\nabla_j u\gamma_k^2\nabla_k u + \sum_{i,k}(\nabla_k\xi_i)\nabla_i u\gamma_k^2\nabla_k u. \end{aligned} \tag{4.11}$$

Note that (4.11) is an equality in $C_b(\mathbb{R}^K)$ since $\nabla_k u$, $(\nabla_k \mathcal{L}_{\xi,\eta}u) \in C_b(\mathbb{R}^K)$ due to Proposition 3.1, (ii).

Recall that $\sum_k \gamma_k^2(\nabla_k u)^2 = |\nabla u|_+^2$. A direct computation shows that

$$\frac{1}{2}\nabla_j|\nabla u|_+^2 = \sum_k \nabla^2_{kj}u\gamma_k^2(\nabla_k u),$$

$$\sum_{i,j,k}a_{ij}(\nabla^3_{kij}u)\gamma_k^2\nabla_k u = 1/2\sum_{i,j,k}a_{ij}\nabla^2_{ij}(\gamma_k^2(\nabla_k u)^2) - \sum_{i,j,k}\gamma_k^2 a_{ij}(\nabla^2_{ik}u)(\nabla^2_{jk}u).$$

Therefore one can rewrite (4.11) as follows

$$\frac{d}{dt}|\nabla u|_+^2(t) = -\mathcal{L}_{\xi,\eta}|\nabla u|_+^2(t) + 2F(t), \tag{4.12}$$

where

$$\begin{aligned}F(t) &= \sum_{i,j,k}((\nabla_k a_{ij})\eta_i\nabla_j u\gamma_k^2\nabla_k u + (\nabla_k\eta_i)a_{ij}\nabla_j u\gamma_k^2\nabla_k u) + \sum_{i,k}(\nabla_k\xi_i)\nabla_i u\gamma_k^2\nabla_k u \\ &+ \sum_{i,j,k}(\nabla_k a_{ij})\nabla^2_{ij}u\gamma_k^2\nabla_k u - \sum_{i,j,k}\gamma_k^2 a_{ij}(\nabla^2_{ik}u)(\nabla^2_{jk}u).\end{aligned}$$

We claim that $|\nabla u|_+^2 \in \mathcal{D}$, where $\mathcal{D}$ is as in Proposition 4.2.1. Indeed, by Proposition 3.1, (iii) we have $|\nabla u|_+^2(t) \in \cap_{p\geq 1}W^{2,p}_{\mathrm{loc}} \cap UC_b^1$ for all $t \geq 0$. Moreover, Proposition 4.2.1, (iii) implies that $F(t) \in C_b$ for all $t > 0$. Since the function $t \to u(t)$ is analytic $(0,\infty) \to C_b^1$ (Proposition 4.2.1, (ii)), we conclude that $\frac{d}{dt}|\nabla u|_+^2(t) \in C_b$ for all $t > 0$. Hence, (4.12) yields

$$-\left(\mathcal{L}_{\xi,\eta}\right)|\nabla u|_+^2(t) = \frac{d}{dt}|\nabla u|_+^2(t) - 2F(t) \in C_b(\mathbb{R}^K),\ t > 0.$$

The definition of $\mathcal{D}$ implies that $|\nabla u|_+^2 \in \mathcal{D}$ and the claim is proved.

It follows from from Proposition 4.2.1, (i) that $|\nabla u|_+^2$ is the classical solution to the non-homogeneous problem for the operator $\mathcal{L}_{\xi,\eta} \restriction \mathcal{D}$. Furthermore, since $t \mapsto u(t)$, $t \mapsto \nabla_k u(t)$, $t \mapsto \mathcal{L}_{\xi,\eta} u(t)$ are continuous functions $[0, \infty) \to C_b$, $k = 1, \ldots, K$, the function $t \mapsto \nabla^2_{km} u(t)$ is continuous. Hence, F is a continuous function $(0, \infty) \to C_b$. Proposition 4.3.1 implies that

$$|\nabla u|_+^2(t) = U(t)|\nabla f|_+^2 + 2\int_0^t U(t-s)F(s)ds,$$

where $U(t)$, $t \geq 0$ is the positive analytic semigroup on C_b, generated by the operator $-\mathcal{L}_{\xi,\eta} \restriction_{\mathcal{D}}$ (Proposition 4.2.1, (i))

The first assumption of the proposition implies that

$$\sum_{k,j} (\nabla_k \xi_i)(\nabla_j u)\gamma_k^2 \nabla_k u \leq c_+ |\nabla u|_+^2.$$

Next we estimate the terms in the expression for F, containing $\nabla_k a_{ij}$. For an arbitrary symmetric matrix $(b_{ij})_{i,j=1}^K$ and any vector $g \in \mathbb{R}^K$ the following inequality holds:

$$\begin{aligned}
&\sum_{i,j,k} (\nabla_k a_{ij}) b_{ij} \gamma_k^2 g_k \\
\leq &\left[\sum_{i,j,l} \gamma_l^2 a_{ij} b_{il} b_{lj}\right]^{\frac{1}{2}} \left[\sum_{i,j,l} \gamma_l^{-2} a^{ij} \left(\sum_k (\nabla_k a_{il})\gamma_k^2 g_k\right)\left(\sum_m (\nabla_m a_{jl})\gamma_m^2 g_m\right)\right]^{\frac{1}{2}} \\
\leq &\left[\sum_{i,j,l} \gamma_l^2 a_{ij} b_{il} b_{lj}\right]^{\frac{1}{2}} |g|_+ \nu_K. \qquad (4.13)
\end{aligned}$$

In order to derive (4.13) we have applied the Cauchy-Schwarz inequality and used definition (4.2) of ν_K. From the boundedness of ν_K (the second assumption of the proposition) we conclude

$$\begin{aligned}
&\sum_{i,j,k} (\nabla_k a_{ij})(\nabla^2_{ij} u)\gamma_k^2 (\nabla_k u) \leq \nu_K |\nabla u|_+ \Big(\sum_{i,j,k} \gamma_k^2 a_{ij} \nabla^2_{ik} u \nabla^2_{jk} u\Big)^{\frac{1}{2}} \\
\leq &(c_+^2/4)|\nabla u|_+^2 + \sum_{i,j,k} \gamma_k^2 a_{ij} \nabla^2_{ik} u \nabla^2_{jk} u
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i,j,k} (\nabla_k a_{ij})\eta_i \nabla_j u \gamma_k^2 \nabla_k u \leq \nu_K |\nabla u|_+ \Big(\sum_{i,j,k} \gamma_k^2 a_{ij} \eta_i (\nabla_k u)\eta_j (\nabla_k u)\Big)^{\frac{1}{2}} \\
\leq &c_+ |\eta|_a |\nabla u|_+^2.
\end{aligned}$$

Thus, it remains to estimate the term in the expression for F, which contains $\nabla_k\eta_i$. Let A and $\nabla\eta$ be the operators in $\mathbb{R}^K$ associated with the matrices $(a_{ij})_{i,j=1}^K$ and $(\nabla_k\eta_i)_{i,k=1}^K$ respectively, and $\widehat{T}$ be the operator defined by the diagonal matrix $(\delta_{jk}\gamma_k)_{k=1}^K$. Then one obtains

$$\sum_{i,j,k} a_{ij}(\nabla_k\eta_i)(\nabla_j u)\gamma_k^2(\nabla_k u) = (\widehat{T}\nabla u,\, (\widehat{T}(\nabla\eta)AT^{-1})T\nabla u)_0$$
$$\leq |\widehat{T}(\nabla\eta)A\widehat{T}^{-1}|_{K,0}|\widehat{T}\nabla u|_0^2 = |(\nabla\eta)A|_{K,0}|\nabla u|_+^2,$$

where $|\cdot|_{K,0}$ stands for the operator norm $(\mathbb{R}^K, |\cdot|_0) \to (\mathbb{R}^K, |\cdot|_0)$. (Here we used the property that for any matrix $\widehat{W}$ one has $\mathrm{sp}\,(\widehat{W}) = \mathrm{sp}\,(\widehat{T}\widehat{W}\widehat{T}^{-1})$.) It is well-known that, for the operator $\widehat{W}$ in $\mathbb{R}^K$ associated with matrix $(w_{jk})_{j,k=1}^K$, we have $|\widehat{W}|_{K,0}^2 \leq \sup\limits_j \sum\limits_k |w_{jk}|^2$. Therefore,

$$|(\nabla\eta)A|_{K,0}^2 = \sup_j \sum_k \Big(\sum_{i=1}^N a_{ij}(\nabla_k\eta_i)\Big)^2$$
$$\leq \sup_j \Big(\sum_{i,k=1}^N (\nabla_k\eta_i)^2\Big)\Big(\sum_{i=1}^N a_{ij}^2\Big) \leq \Big(\sum_{i=1}^N \bar{c}_i\Big)\Big(\sum_{i,k=1}^N (\nabla_k\eta_i)^2\Big).$$

In order to obtain the last inequality we have made use of **(A2)**. Combining the derived estimates we obtain that

$$\sum_{i,j,k} a_{ij}(\nabla_k\eta_i)\nabla_j u\gamma_k^2\nabla_k u \leq \Big(\sum_{i=1}^N \bar{c}_i\Big)^{\frac{1}{2}}|\nabla\eta|_{HS}|\nabla u|_+^2.$$

Since $|\nabla u|_+^2$ is non-negative and the semigroup $U(t)$ is positive and contractive we have

$$\|\,|\nabla u|_+\|_\infty^2 \leq \|\,|\nabla f|_+\|_\infty^2 + 2C_+\int_0^t \|\,|\nabla u|_+\|_\infty^2(s)ds.$$

Hence the assertion follows from Gronwall's lemma. □

For $d > K$ we set $\eta_j = 0$, $j = K+1, \ldots, d$, and introduce the quantities

$$\xi_{a;k}^{\mu,d} := \sum_{j=1}^d (a_{kj}\xi_j^\mu + \nabla_j a_{kj}),\ k = 1, \ldots, K,$$
$$B_d := \sum_k (\xi_k - \xi_{a;k}^{\mu,d})\nabla_k u + \sum_{j=1}^d \sum_k a_{kj}(\eta_j - \eta_j^\mu)\nabla_k u,$$

For $p \geq 1$ we set $[\nabla u]_{\varepsilon,a}^2 := |\nabla u|_a^2 + \varepsilon^2$ with $\varepsilon > 0$. Set $\chi_\varepsilon := [\nabla u]_{\varepsilon,a}^{p-2}$. We introduce the following quantities:

$$T_{\varepsilon,a} := \| \, [\nabla u]_{\varepsilon,a}^{p-1} |\nabla u|_a \|_2^2 \,,$$
$$J_{\varepsilon,a} := \| \, [\nabla u]_{\varepsilon,a}^{p-3} |\nabla |\nabla u|_a^2|_a \|_2^2 \,,$$
$$I_{\varepsilon,a} := \sum_{i,j,k,l} \langle \chi_\varepsilon^2 a_{ij} \nabla_{jk}^2 u, \, a_{kl} \nabla_{li}^2 u \rangle \,.$$

Note that $I_{\varepsilon,a} = \|\chi_\varepsilon |AD^2 u|_{HS(a)}\|_2^2$ and $(p-1)^2 J_{\varepsilon,a} = 4 \| \, |\nabla [\nabla u]_{\varepsilon,a}^{p-1}|_a \|_2^2$.

Lemma 4.3.2. *Let u be the solution to* (4.4). *Then*

$$\|\chi_\varepsilon \frac{du}{dt}\|_2^2 + \frac{1}{p-1}\frac{d}{dt} \| \, [\nabla u]_{\varepsilon,a}\|_{2p-2}^{2p-2} \leq 2\|\chi_\varepsilon B_d\|_2^2 + 2(p-2)^2 J_{\varepsilon,a}. \tag{4.14}$$

Proof. It follows from (4.4) that

$$\Big\langle u_t - \sum_{k,j} a_{kj} \nabla_{kj}^2 u, \, \chi_\varepsilon^2 u_t \Big\rangle = \Big\langle \sum_{k,j} a_{kj} \eta_k \nabla_j u + \sum_j \xi_j \nabla_j u, \, \chi_\varepsilon^2 u_t \Big\rangle.$$

Integration by parts yields

$$-\sum_{k,j} \langle a_{kj} \nabla_{kj}^2 u, \, \chi_\varepsilon^2 u_t \rangle = \sum_{k=1}^d \sum_j \langle a_{kj} \nabla_j u, \, (\nabla_k + \eta_k^\mu) \chi_\varepsilon^2 u_t \rangle + \sum_j \langle \xi_{a;j}^{\mu,d} \nabla_j u, \, \chi_\varepsilon^2 u_t \rangle.$$

Hence we obtain

$$\begin{aligned}
&\|\chi_\varepsilon u_t\|_2^2 + \frac{1}{2p-2}\frac{d}{dt} \| \, [\nabla u]_{\varepsilon,a}\|_{2p-2}^{2p-2} \\
&= \sum_{k=1}^d \sum_j \langle a_{kj} (\eta_k - \eta_k^\mu) \nabla_j u, \, \chi_\varepsilon^2 u_t \rangle + \sum_j \langle (\xi_j - \xi_{a;j}^{\mu,d}) \nabla_j u, \, \chi_\varepsilon^2 u_t \rangle \\
&- \sum_{k,j} \langle a_{kj} \nabla_j u \nabla_k \chi_\varepsilon^2, \, u_t \rangle \leq \frac{1}{2} \|\chi_\varepsilon u_t\|_2^2 + \|\chi_\varepsilon B_d\|_2^2 + (p-2)^2 J_{\varepsilon,a}.
\end{aligned} \tag{4.15}$$

The last inequality in (4.15) follows from the estimate

$$4\langle \, |(\nabla u, \, \nabla \chi_\varepsilon)_a|^2 \rangle \leq 4 \| \, [\nabla u]_{\varepsilon,a} |\nabla \chi_\varepsilon|_a \|_2^2 = (p-2)^2 J_{\varepsilon,a}, \tag{4.16}$$

and (4.15) implies the assertion. $\square$

Lemma 4.3.3. *Let u be the solution to* (4.4). *Then for any $\delta > 0$ we have*

$$\begin{aligned}
&\|\chi_\varepsilon |\nabla u|_a\|_2^2 + \frac{\delta}{p-1}\frac{d}{dt} \| \, [\nabla u]_{\varepsilon,a}\|_{2p-2}^{2p-2} \\
&\leq 3\delta \|\chi_\varepsilon B_d\|_2^2 + 3\delta (p-2)^2 J_{\varepsilon,a} + \frac{3}{4\delta} \|\chi_\varepsilon u\|_2^2
\end{aligned} \tag{4.17}$$

Proof. Let ψ_ε be defined by $\psi_\varepsilon := -\sum_{k=1}^d \sum_j (\nabla_k + \beta_k^\mu)(\chi_\varepsilon^2 a_{kj} \nabla_j u)$. It follows from (4.4) that

$$\psi_\varepsilon = \chi_\varepsilon^2 \Big[-u_t + \sum_{j=1}^d \sum_k a_{kj}(\eta_j - \eta_j^\mu)\nabla_k u + \sum_k (\xi_k - \xi_{a;k}^{\mu,d})\nabla_k u \Big]$$
$$- \sum_{k,j} a_{kj}(\nabla_k \chi_\varepsilon^2)\nabla_j u \equiv \chi_\varepsilon^2 (B_d - u_t) - 2\chi_\varepsilon (\nabla u,\, \nabla \chi_\varepsilon)_a. \tag{4.18}$$

Integration by parts yields

$$\|\chi_\varepsilon |\nabla u|_a\|_2^2 = \langle u,\, \psi_\varepsilon \rangle = \langle u,\, \chi_\varepsilon^2 (B_d - u_t) - 2\chi_\varepsilon (\nabla u,\, \nabla \chi_\varepsilon)_a \rangle \tag{4.19}$$

We estimate the right-hand side of (4.19) as follows.

$$\langle u,\, \chi_\varepsilon^2 B_d \rangle| \le \delta \|\chi_\varepsilon B_d\|_2^2 + \frac{1}{4\delta}\|\chi_\varepsilon u\|_2^2,$$
$$2|\langle u,\, (\nabla u,\, \nabla \chi_\varepsilon)_a \rangle| \le \delta (p-2)^2 J_{\varepsilon,a} + \frac{1}{4\delta}\|\chi_\varepsilon u\|_2^2,$$
$$|\langle u,\, \chi_\varepsilon^2 u_t \rangle| \le \delta \|\chi_\varepsilon u_t\|_2^2 + \frac{1}{4\delta}\|\chi_\varepsilon u\|_2^2,$$

(where we have used (4.16) in the second term). Applying Lemma 4.1 we complete the proof. □

We introduce the following quantities.

$$\Upsilon_d := |\eta^\mu|_a + |\eta|_a + v_d,$$
$$\Xi_d^2 := |\nabla u|_+^2 \sum_{k=1}^K \gamma_k^{-2} (\xi_k - \xi_{a;k}^{\mu,d})^2,$$
$$\Xi^2 := L^1\text{-} \lim_{d\to\infty} \Xi_d^2 = |\nabla u|_+^2 \sum_{k=1}^K \gamma_k^{-2}(\xi_k - \xi_{a;k}^\mu)^2 = |\nabla u|_+^2 |\xi - P_K \xi_a^\mu|_-^2,$$

where v_d is as in (4.1) and the limit in d exists due to (A4).

Lemma 4.3.4. *Let $p \in (3 - \frac{3}{1+\sqrt{1+3\varepsilon_0}}, \frac{2}{\varepsilon_0})$. Then there exist positive constants $K(\varepsilon_0, p)$ and $r(p)$ such that*

$$r(p)\frac{d}{dt}\|[\nabla u]_{\varepsilon,a}\|_{2p-2}^{2p-2} + K(\varepsilon_0, p) J_{\varepsilon,a}$$
$$\le C_{\varepsilon_0,p}\Big[\|\chi_\varepsilon \Xi\|_2^2 + \sup_d \|\chi_\varepsilon |\nabla u|_a \Upsilon_d\|_2^2\Big] + C_p \|\chi_\varepsilon u\|_2^2.$$

If $p < 2$ the same estimate holds for $J_{\varepsilon,a}$ replaced by $I_{\varepsilon,a}$.

Proof. It follows from (4.4) that

$$\langle u_t,\, \psi_\varepsilon\rangle - \sum_{k,l}\langle a_{kl}\nabla^2_{kl}u,\, \psi_\varepsilon\rangle = \sum_{k,l}\langle a_{kl}\eta_l\nabla_l u,\, \psi_\varepsilon\rangle + \sum_l\langle \xi_l\nabla_l u,\, \psi_\varepsilon\rangle, \tag{4.20}$$

where the function ψ_ε was defined in Lemma 4.3.3. It is easy to see that the second term in the left-hand side of (4.20) equals

$$-\sum_{k=1}^d\sum_l\langle\nabla_k(a_{kl}\nabla_l u),\, \psi_\varepsilon\rangle + \sum_{k=1}^d\sum_l\langle(\nabla_k a_{kl})\nabla_l u,\, \psi_\varepsilon\rangle.$$

The key point is to evaluate the first term in the above expression. Successive integration by parts and a straightforward computation give

$$\begin{aligned}
&\sum_{i,k=1}^d\sum_{j,l}\big\langle\nabla_k(a_{kl}\nabla_l u),\, (\nabla_i+\beta_i^\mu)(\chi_\varepsilon^2 a_{ij}\nabla_j u)\big\rangle\\
=&-\sum_{i,k=1}^d\sum_{j,l}\big\langle\nabla_i\nabla_k(a_{kl}\nabla_l u),\, \chi_\varepsilon^2 a_{ij}\nabla_j u\big\rangle\\
=&\sum_{i,k=1}^d\sum_{j,l}\big[\langle\nabla_i(a_{kl}\nabla_l u),\, (\nabla_k+\eta_k^\mu)(\chi_\varepsilon^2 a_{ij}\nabla_j u)\rangle + \langle\nabla_i(a_{kl}\nabla_l u),\, \xi_k^\mu\chi_\varepsilon^2 a_{ij}\nabla_j u\rangle\big]\\
=&S_1+S_2,
\end{aligned}$$

with

$$S_1 = \sum_{i,k=1}^d\sum_{j,l}\langle\nabla_i(a_{kl}\nabla_l u),\, (\nabla_k+\eta_k^\mu)(\chi_\varepsilon^2 a_{ij}\nabla_j u)\rangle$$

and

$$S_2 = \sum_{i,k=1}^d\sum_{j,l}\langle\nabla_i(a_{kl}\nabla_l u),\, \xi_k^\mu\chi_\varepsilon^2 a_{ij}\nabla_j u\rangle.$$

It is easy to see that

$$\begin{aligned}
S_1 = I_{\varepsilon,a} &+ \sum_{i,j,k,l}\big[2\langle\chi_\varepsilon^2 a_{ij}\nabla^2_{kj}u,\, (\nabla_i a_{kl})\nabla_l u\rangle + \langle a_{ij}\nabla_j u\nabla_k(\chi_\varepsilon^2),\, \nabla_i(a_{kl}\nabla_l u)\rangle\big]\\
&+\sum_{i,k=1}^d\sum_{j,l}\big[\langle\chi_\varepsilon^2(\nabla_i a_{kj})\nabla_j u,\, (\nabla_k a_{il})\nabla_l u\rangle + \langle\chi_\varepsilon^2 a_{ij}\nabla_j u\,\eta_k^\mu,\, \nabla_i(a_{kl}\nabla_l u)\rangle\big].
\end{aligned}$$

We transform S_2 as follows

$$S_2 = \sum_{i,k=1}^{d} \sum_{j,l} \langle \chi_\varepsilon^2 \nabla_i (a_{kl} \nabla_l u),\, \xi_k^\mu a_{ij} \nabla_j u \rangle = \lim_m \sum_{i,k=1}^{d} \sum_{j,l} \langle \chi_\varepsilon^2 \nabla_i (a_{kl} \nabla_l u),\, \hat{\xi}_k^m a_{ij} \nabla_j u \rangle$$

$$= \lim_m \sum_{i,k=1}^{d} \sum_{j,l} \Big(\langle \hat{\xi}_k^m a_{kl} \nabla_l u,\, (\nabla_i + \beta_i^\mu) \chi_\varepsilon^2 a_{ij} \nabla_j u \rangle - \langle (\nabla_i \hat{\xi}_k^m) a_{kl} \nabla_l u,\, \chi_\varepsilon^2 a_{ij} \nabla_j u \rangle \Big)$$

$$= \sum_{k=1}^{d} \sum_{l} \langle a_{kl} \xi_k^\mu \nabla_l u,\, \psi_\varepsilon \rangle - \lim_m \sum_{i,k=1}^{d} \sum_{j,l} \langle (\nabla_i \hat{\xi}_k^m) a_{kl} \nabla_l u,\, \chi_\varepsilon^2 a_{ij} \nabla_j u \rangle.$$

Thus we have

$$-\sum_{k,l} \langle a_{kl} \nabla_{kl}^2 u,\, \psi_\varepsilon \rangle = S_1 + \sum_{l} \langle (\xi_l - \xi_{a;l}^{\mu,d}) \nabla_l u,\, \psi_\varepsilon \rangle$$
$$- \lim_m \sum_{i,k=1}^{d} \sum_{j,l} \langle (\nabla_i \hat{\xi}_k^m) a_{kl} \nabla_l u,\, \chi_\varepsilon^2 a_{ij} \nabla_j u \rangle.$$

We observe that

$$\langle u_t,\, \psi_\varepsilon \rangle = \frac{1}{2} \Big\langle \chi_\varepsilon^2 \frac{d}{dt} |\nabla u|_a^2 \Big\rangle = \frac{1}{2p-2} \frac{d}{dt} \| [\nabla u]_{\varepsilon,a} \|_{2p-2}^{2p-2}.$$

So we infer from (4.20) that

$$\begin{aligned}
&\frac{1}{2p-2} \frac{d}{dt} \| [\nabla u]_{\varepsilon,a} \|_{2p-2}^{2p-2} + I_{\varepsilon,a} + \frac{p-2}{2} J_{\varepsilon,a} \\
&= \Big[\frac{p-2}{2} J_{\varepsilon,a} - \sum_{i,j,k,l} \langle a_{ij} \nabla_j u \nabla_k (\chi_\varepsilon^2),\, \nabla_i (a_{kl} \nabla_l u) \rangle \Big] \\
&- \sum_{i,j,k,l} \big[2 \langle \chi_\varepsilon^2 a_{ij} \nabla_{kj}^2 u,\, (\nabla_i a_{kl}) \nabla_l u \rangle + \langle a_{ij} \nabla_j u \nabla_k (\chi_\varepsilon^2),\, \nabla_i (a_{kl} \nabla_l u) \rangle \big] \\
&- \sum_{i,k=1}^{d} \sum_{j,l} \big[\langle \chi_\varepsilon^2 (\nabla_i a_{kj}) \nabla_j u,\, (\nabla_k a_{il}) \nabla_l u \rangle + \langle \chi_\varepsilon^2 a_{ij} \eta_k^\mu \nabla_j u,\, \nabla_i (a_{kl} \nabla_l u) \rangle \big] \\
&+ \lim_m \sum_{i,k=1}^{d} \sum_{j,l} \langle (\nabla_i \hat{\xi}_k^m) a_{kl} \nabla_l u,\, \chi_\varepsilon^2 a_{ij} \nabla_j u \rangle \\
&- \sum_{k,l} \langle a_{kl} \eta_l \nabla_k u,\, \psi_\varepsilon \rangle - \sum_{k} \langle (\xi_k - \xi_{a;k}^{\mu,d}) \nabla_k u,\, \psi_\varepsilon \rangle.
\end{aligned} \tag{4.21}$$

In order to estimate the right-hand side of (4.21) we use the following inequalities:

$$\Big|\sum_{i,k=1}^{d}\sum_{j,l}(\nabla_i a_{kj})\nabla_j u(\nabla_k a_{il})\nabla_l u\Big| \le v_d^2|\nabla u|_a^2, \tag{4.22}$$

$$\sum_{i,j,k,l} a_{ij}(\nabla_{kj}^2 u)(\nabla_i a_{kl})\nabla_l u \le \delta|AD^2u|^2_{HS(a)} + C_\delta v_d^2|\nabla u|_a^2, \tag{4.23}$$

$$\Big|\sum_{i,j,k,l}\langle(a_{ij}\nabla_j u)\nabla_k\chi_\varepsilon^2,\ \nabla_i(a_{kl}\nabla_l u)\rangle - \frac{p-2}{2}J_{\varepsilon,a}\Big|$$
$$\le \delta(p-2)^2 J_{\varepsilon,a} + C_\delta\|\chi_\varepsilon|\nabla u|_a v_d\|_2^2, \tag{4.24}$$

for any positive δ. (Recall that $(h,\,g)_a := \sum_{k,j\ge 1} a_{kj}h_j g_k$.) The estimates (4.22), (4.23) are immediate from the definitions and the Cauchy inequality. We postpone the proof of (4.24) to section 4.5. It follows from assumption (ii(b)) of Theorem 4.1.2 that the limit in m and d in the third term from the end in (4.21) does not exceed

$$\varepsilon_0\sum_{i,j,k,l}\langle a_{li}\nabla_i(\chi_\varepsilon\nabla_j u),\ a_{jk}\nabla_k(\chi_\varepsilon\nabla_l u)\rangle + c(\varepsilon_0)\sum_{k,j}\langle\chi_\varepsilon^2 a_{kj}\nabla_k u\nabla_j u\rangle$$
$$\le\varepsilon_0\left(I_{\varepsilon,a} + \frac{(p-2)^2}{4}J_{\varepsilon,a} + |p-2|\sqrt{I_{\varepsilon,a}J_{\varepsilon,a}}\right) + c(\varepsilon_0)\|\chi_\varepsilon|\nabla u|_a\|_2^2. \tag{4.25}$$

Here we used the fact that

$$\sum_{i,j,k,l}\langle a_{jk}(\nabla_i\chi_\varepsilon)\nabla_j u,\ a_{li}(\nabla_k\chi_\varepsilon)\nabla_l u\rangle = \|(\nabla u,\ \nabla\chi_\varepsilon)_a\|_2^2,$$

and applied (4.16).

In order to estimate the terms containing ψ_ε we use (4.16), (4.18) and the inequality

$$B_d^2 \le 2(\Xi_d^2 + \Upsilon_d^2|\nabla u|_a^2).$$

Hence, for any positive δ we have

$$|\sum_{k,l}\langle a_{kl}\eta_l\nabla_k u,\ \psi_\varepsilon\rangle + \sum_k\langle(\xi_k - \xi^\mu_{a;k})\nabla_k u,\ \psi_\varepsilon\rangle|$$
$$\le\delta(p-2)^2 J_{\varepsilon,a} + \delta\|\chi_\varepsilon u_t\|_2^2 + C_\delta\|\chi_\varepsilon\Xi_d\|_2^2 + C_\delta\|\chi_\varepsilon|\nabla u|_a\Upsilon_d\|_2^2$$

Making use of Lemma 4.3.2 we arrive at

$$|\sum_{k,l}\langle a_{kl}\eta_l\nabla_k u,\ \psi_\varepsilon\rangle + \sum_k\langle(\xi_k - \xi^\mu_{a;k})\nabla_k u,\ \psi_\varepsilon\rangle|$$
$$\le 2\delta(p-2)^2 J_{\varepsilon,a} - \frac{\delta}{p-1}\frac{d}{dt}\|\,[\nabla u]_{\varepsilon,a}\|_{2p-2}^{2p-2} + C_\delta\|\chi_\varepsilon\Xi_d\|_2^2 + C_\delta\|\chi_\varepsilon|\nabla u|_a\Upsilon_d\|_2^2. \tag{4.26}$$

Combining (4.22)-(4.26) and using Lemma 4.3.3 (to estimate $\|\chi_\varepsilon|\nabla u|_a\|_2^2$ in (4.25)) we get

$$\begin{aligned}
&\frac{1}{p-1}\left(\frac{1}{2}+\delta+c(\varepsilon_0)\delta\right)\frac{d}{dt}\|[\nabla u]_{\varepsilon,a}\|_{2p-2}^{2p-2}\\
&+(1-\varepsilon_0)I_{\varepsilon,a}+\frac{2(p-2)-\varepsilon_0(p-2)^2}{4}J_{\varepsilon,a}-\varepsilon_0|p-2|\sqrt{I_{\varepsilon,a}J_{\varepsilon,a}}\\
&\leq \delta I_{\varepsilon,a}+3\delta(1+c(\varepsilon_0))(p-2)^2 J_{\varepsilon,a}\\
&+C_{\delta,\varepsilon_0}\sup_d\|\chi_\varepsilon|\nabla u|_a\Upsilon_d\|_2^2+C_{\delta,\varepsilon_0}\lim_{d\to\infty}\|\chi_\varepsilon\Xi_d\|_2^2+C_\delta\|\chi_\varepsilon u\|_2^2. \qquad (4.27)
\end{aligned}$$

Note that $\Xi_d \to \Xi$ as $d\to\infty$ in L^2 due to **(A4)**. Now applying Lemma 4.5.2 (see section 4.5 below) we complete the proof. □

Proof of Proposition 4.2.3. First let $p\geq 2$. Lemma 4.5.1 below states that

$$\|\,|\nabla u|_a\|_{2p}^{2p}\leq 2\|u\|_\infty^2\|\,|\nabla u|_a^{p-2}\sum_{k=1}^{d}\sum_j(\nabla_k+\beta_k^\mu)(a_{kj}\nabla_j u)\|_2^2+2(p-1)^2\|u\|_\infty^2 J_{0,a}.$$

(Here and below $J_{0,a}:=\lim_{\varepsilon\to 0}J_{\varepsilon,a}=4(p-1)^{-2}\|\,|\nabla|\nabla u|_a^{p-1}|_a^2\|_2^2$.) Making use of equation (4.4) and the maximum principle we get

$$\|\,|\nabla u|_a\|_{2p}^{2p}\leq 2\|f\|_\infty^2\Big[\|\,|\nabla u|_a^{p-2}u_t\|_2^2+\||\nabla u|_a^{p-2}B_d\|_2^2\Big]+2(p-1)^2\|f\|_\infty^2 J_{0,a}. \quad (4.28)$$

Observe that in Lemma 4.3.2 one can pass to the limit as $\varepsilon\to 0$ provided $p\geq 2$:

$$\|\,|\nabla u|_a^{p-2}u_t\|_2^2\leq 2\|\,|\nabla u|_a^{p-2}B_d\|_2^2+2(p-2)^2J_{0,a}-\frac{1}{p-1}\frac{d}{dt}\|\,|\nabla u|_a\|_{2p-2}^{2p-2}.$$

Hence, estimating $\|\,|\nabla u|_a^{p-2}u_t\|_2^2$ in (4.28) we conclude that

$$\|\,|\nabla u|_a\|_{2p}^{2p}\leq C_p\|f\|_\infty^2(\||\nabla u|_a^{p-2}B_d\|_2^2+J_{0,a})-4\frac{\|f\|_\infty^2}{p-1}\frac{d}{dt}\|\,|\nabla u|_a\|_{2p-2}^{2p-2}. \quad (4.29)$$

Next we note that

$$\limsup_{d\to\infty}\||\nabla u|_a^{p-2}B_d\|_2^2\leq 2\||\nabla u|_a^{p-2}\Xi\|_2^2+2\sup_d\||\nabla u|_a^{p-1}\Upsilon_d\|_2^2.$$

It is easy to see that we can also pass to the limit as $\varepsilon\to 0$ in Lemma 4.3.4. Applying Lemma 4.3.4 to (4.29) we obtain

$$\begin{aligned}
&\kappa_1(\varepsilon_0,p)\|f\|_\infty^2\frac{d}{dt}\|\,|\nabla u|_a\|_{2p-2}^{2p-2}+\kappa_2(\varepsilon_0,p)\|\,|\nabla u|_a\|_{2p}^{2p}\\
&\leq C_{\varepsilon_0,p}\|f\|_\infty^2\Big(\||\nabla u|_a^{p-2}\Xi\|_2^2+\||\nabla u|_a^{p-2}u\|_2^2+\sup_d\||\nabla u|_a^{p-1}\Upsilon_d\|_2^2\Big), \qquad (4.30)
\end{aligned}$$

with some positive $\kappa_1(\varepsilon_0, p)$ and $\kappa_2(\varepsilon_0, p)$. We estimate the right-hand side of (4.30) from above by

$$\delta \| \, |\nabla u|_a \|_{2p}^{2p} + C_{p,\varepsilon_0,\delta} \|f\|_\infty^p \Big(\|f\|_\infty^p + \|\Xi\|_p^p \Big) + C_{p,\varepsilon_0,\delta} \|f\|_\infty^{2p} \sup_d \|\Upsilon_d\|_{2p}^{2p},$$

for any positive δ. Choosing δ small enough we arrive at the inequality

$$\begin{aligned} &\kappa_1(\varepsilon_0, p) \|f\|_\infty^2 \frac{d}{dt} \| \, |\nabla u|_a \|_{2p-2}^{2p-2} + \kappa_2(\varepsilon_0, p) \| \, |\nabla u|_a \|_{2p}^{2p} \\ &\le C_{p,\varepsilon_0} \|f\|_\infty^p \|\Xi\|_p^p + C_{p,\varepsilon_0} \|f\|_\infty^{2p} \Big(\sup_d \|\Upsilon_d\|_{2p}^{2p} + 1 \Big). \end{aligned} \tag{4.31}$$

Now we assume that $p < 2$. As in the case $p \geq 2$ we employ Lemma 4.5.1, equation (4.4), the maximum principle and Lemma 4.3.2. Then

$$\begin{aligned} T_{\varepsilon,a} &\le \varepsilon^{2p-2} \|u\|_2^2 + 4(\|u\|_\infty^2 + \varepsilon^2) \Big[\|\chi_\varepsilon u_t\|_2^2 + \|\chi_\varepsilon B_d\|_2^2 \Big] + 2(p-1)^2 \|u\|_\infty^2 J_{\varepsilon,a} \\ &\le \varepsilon^{2p-2} \|f\|_\infty^2 + C_p(\|f\|_\infty^2 + \varepsilon^2)(\|\chi_\varepsilon B_d\|_2^2 + J_{\varepsilon,a}) \\ &- 4 \frac{\|f\|_\infty^2 + \varepsilon^2}{p-1} \frac{d}{dt} \| \, [\nabla u]_{\varepsilon,a} \|_{2p-2}^{2p-2}. \end{aligned}$$

Setting $\varepsilon := \|f\|_\infty$, passing to the limit as $d \to \infty$ and employing Lemma 4.3.4 we derive the estimate

$$\begin{aligned} &\kappa_1(\varepsilon_0, p) \|f\|_\infty^2 \frac{d}{dt} \| \, [\nabla u]_{\varepsilon,a} \|_{2p-2}^{2p-2} + \kappa_2(\varepsilon_0, p) T_{\varepsilon,a} \\ &\le C_{\varepsilon_0,p} \|f\|_\infty^2 \big[\|\chi_\varepsilon \Xi\|_2^2 + \sup_d \|\chi_\varepsilon |\nabla u|_a \Upsilon_d\|_2^2 + \|\chi_\varepsilon u\|_2^2 + \|f\|_\infty^{2p-2} \big]. \end{aligned} \tag{4.32}$$

Observe that $\chi_\varepsilon \le \varepsilon^{p-2}$ and $(\chi_\varepsilon |\nabla v|_a)^{p'} \le |\nabla v|_{a,\varepsilon}^{p-1} |\nabla v|_a$. The Young inequality implies that

$$\|\chi_\varepsilon |\nabla u|_a \phi\|_2^2 \le \delta T_{\varepsilon,a} + C_{p,\delta} \|\phi\|_{2p}^{2p}, \tag{4.33}$$

for all $\phi \in L^{2p}$ and any positive δ. We substitute (4.33) into (4.32), choose $\delta > 0$ small enough and obtain

$$\begin{aligned} &\kappa_1(\varepsilon_0, p) \|f\|_\infty^2 \frac{d}{dt} \| \, [\nabla u]_{\varepsilon,a} \|_{2p-2}^{2p-2} + \kappa_2(\varepsilon_0, p) T_{\varepsilon,a} \\ &\le C_{\varepsilon_0,p} \Big(\|f\|_\infty^{2p-2} \|\Xi\|_2^2 + \|f\|_\infty^{2p} \big(\sup_d \|\Upsilon_d\|_{2p}^{2p} + 1 \big) \Big). \end{aligned} \tag{4.34}$$

In order to complete the proof of (4.6) we apply Proposition 4.2.2 to estimate $\|\Xi\|_s^s$ in (4.31) and (4.34) and integrate the derived inequalities from 0 to t.

When $p = 2$ we apply the Hölder inequality, Proposition 4.2.2 and (4.6) to (4.27) in order to obtain (4.7).

If $p < 2$ it follows from the Young inequality that

$$\| |AD^2 u|_{HS(a)} f \|_p^p \le \varepsilon^{-p} \| [\nabla u]_{\varepsilon,a} \|_{2p}^{2p} + C_p \varepsilon^{2-p} I_{\varepsilon,a}. \tag{4.35}$$

We employ the Young and Hölder inequalities to estimate the first term in the right-hand side of (4.35).

$$\| [\nabla u]_{\varepsilon,a} \|_{2p}^{2p} \le T_{\varepsilon,a} + \varepsilon^2 \| [\nabla u]_{\varepsilon,a}^{p-1} \|_2^2 \le T_{\varepsilon,a} + (1/2) \| [\nabla u]_{\varepsilon,a} \|_{2p}^{2p} + C_p \varepsilon^{2p}. \tag{4.36}$$

We take $\varepsilon := \|f\|_\infty$. Now making successive use of (4.35), (4.36), Lemma 4.3.4, (4.27), (4.34) and Proposition 4.2.2 one completes the proof. □

Proof of Proposition 4.2.4. It follows from (4.4) that

$$\Big\langle u_t - \sum_{k,j} a_{kj} \nabla_{kj}^2 u,\ u \Big\rangle = \Big\langle \sum_{k,j} a_{kj} \eta_k \nabla_j u + \sum_j \xi_j \nabla_j u,\ u \Big\rangle.$$

Integration by parts yields

$$- \sum_{k,j} \langle a_{kj} \nabla_{kj}^2 u,\ u \rangle = \sum_{k=1}^{d} \sum_j \langle a_{kj} \nabla_j u,\ (\nabla_k + \eta_k^\mu) u \rangle + \sum_j \langle \xi_{a;j}^{\mu,d} \nabla_j u,\ u \rangle.$$

Making use of the maximum principle, the Hölder and Cauchy inequalities we obtain the estimate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \|u\|_2^2 + \| |\nabla u|_a \|_2^2 \\
&= \sum_{k=1}^{d} \sum_j \langle a_{kj} (\eta_k - \eta_k^\mu) \nabla_j u,\ u \rangle + \sum_j \langle (\xi_j - \xi_{a;j}^{\mu,d}) u \rangle \\
&\le \frac{1}{2} \| |\nabla u|_a \|_2^2 + \frac{1}{2} \|f\|_\infty^2 \| |\eta - \eta^\mu|_a \|_2^2 + \|f\|_\infty \| |\nabla u|_+ \|_\infty \| |\xi - P_K \xi_a^\mu|_- \|_1.
\end{aligned} \tag{4.37}$$

We apply Proposition 4.2.2 to the right-hand side of (4.37) and integrate the obtained inequality from 0 to t. This completes the proof. □

4.4 Example

Let $X = \mathbb{R}^{\mathbb{N}}$, $H_0 = l^2$, $H_+ = l^2_{\gamma_k}$ and $H_- = l^2_{\gamma_k^{-1}}$ with $(\gamma_k)_{k\in\mathbb{N}} \subset (0, \infty)$, where $l^2_{\gamma_k}$ and $l^2_{\gamma_k^{-1}}$ are described in section 4.1.

Let $(s_k)_{k\in\mathbb{N}} \subset (0, \infty)$. An operator S in H_0 with the domain $\mathbb{R}^{\text{fin}}$ is defined by its matrix elements: $s_{jk} := \delta_{jk} s_k$, $j, k \in \mathbb{N}$. This operator is positive. Let μ_S stand for the Gaussian measure with correlation operator S. Recall that

$$\mu_S = \prod_{k\ge1} e^{-\frac{x_k^2}{2s_k}} \frac{dx_k}{\sqrt{2\pi s_k}}.$$

Let μ be a probability measure on $\mathbb{R}^\infty$ given by

$$\mu := \prod_{k\geq 1} \frac{1}{m_k\Gamma(m_k/2)} x_k^{m_k} e^{-\frac{x_k^2}{2s_k}} \frac{dx_k}{\sqrt{2s_k}},$$

where $m_k > 0$, $k \in \mathbb{N}$.

It is easy to see that, for $k \in \mathbb{N}$, $\beta_k^\mu(x) = -s_k^{-1}x_k + m_k|x_k|^{-1} = \xi_k^\mu(x) + \eta_k^\mu(x)$ with $\xi_k^\mu(x) := -s_k^{-1}x_k$ and $\eta_k^\mu(x) := m_k|x_k|^{-1}$, $x \in \mathbb{R}^\mathbb{N}$.

For $\delta > 0$ we introduce functions

$$a_{jk}(x) := \delta_{jk}\frac{x_k^2+\delta}{x_k^2+1}, \quad x \in \mathbb{R}^\infty.$$

Note that for every $N \in \mathbb{N}$ the matrix $(a_{jk})_{j,k=1}^N$ is cylindric, smooth and uniformly elliptic.

A direct computation implies that $\xi_{a;k}^\mu = -\frac{(x_k^2+\delta)x_k}{(x_k^2+1)s_k} + \frac{2(1-\delta)x_k}{(x_k^2+1)^2}$, $k \in \mathbb{N}$.

Let $p \geq 1$. For $k \in \mathbb{N}$ we set $m_k := 2k+2p-1$. We choose the sequence $(s_k)_{k\in\mathbb{N}}$ in such way that the series

$$\sum_{k\geq 1} m_k^{\frac{2p-1}{2p}} \left(\frac{\Gamma(m_k/2+1-p)}{\Gamma(m_k/2)}\right)^{\frac{1}{2p}} s_k^{\frac{m_k/2-p}{2p}}$$

is convergent. This is the case if, for example, the sequence $(s_k)_{k\in\mathbb{N}}$ is bounded. Then one can check that $|\eta^\mu - P_N\eta^\mu|_a \to 0$ in $L^{2p} := L^{2p}(\mathbb{R}^\mathbb{N}, \mu)$ as $N \to \infty$.

Computing that

$$\nabla_k a_{kk}(x) = \frac{2\delta x_k}{(1+x_k^2)^2}$$

and using the definitions of v_N and ν_N we see that $v_N \leq 2|1-\delta|$ and $\nu_N \leq \frac{|1-\delta|}{\delta}$ for all $N \in \mathbb{N}$.

We choose $\xi_k^n := -\frac{(x_k^2+\delta)x_k}{(x_k^2+1)s_k} + \frac{2(1-\delta)x_k}{(x_k^2+1)^2}$, $n \in \mathbb{N}$, $k = 1, \ldots, n$. Then condition (i(a)) of Theorem 4.1.2 is satisfied. One can readily see that

$$\nabla_k \xi_k^n = -\frac{1}{s_k}\left(1-(1-\delta)\frac{1-x_k^2}{(1+x_k^2)^2}\right) + \frac{2(1-\delta)(1-x_k^2)^2}{(1+x_k^2)^3}.$$

An elementary analysis of the last expression shows that condition (i(b)) of Theorem 4.1.2 holds for an arbitrary sequence of positive numbers $(s_k)_{k\in\mathbb{N}}$, provided $\delta \in (0, 9]$. one can check directly that the sequence $\hat{\xi}_k^m := -s_k^{-1}x_k$, $m \in \mathbb{N}$, $k = 1, \ldots, m$ satisfies condition (ii(b)) with $\varepsilon_0 = 0$ and $c(\varepsilon_0) = 0$.

Hence, by Theorem 4.1.2 the set $\mathcal{F}C_b^\infty$ is a core of the operator $\mathcal{L}_p$ in L^p for all $p > 3/2$.

4.5 Auxiliary inequalities

Let $v \in \mathcal{F}C_b^{2,u}(\mathbb{R}^K)$ and quantities $T_{\varepsilon,a}$, $I_{\varepsilon,a}$ and $J_{\varepsilon,a}$ be defined as in section 4.3 (with v replacing the solution u of (4.4)). (Recall that then $\chi_\varepsilon = [\nabla v]_{\varepsilon,a}^{p-2}$). We use the same summation convention as in section 4.3.

Below we present several estimates which are used in the proof of Proposition 4.2.3. It is noteworthy that Lemma 4.5.1 below is an extension of the Gagliardo–Nirenberg inequality to the case when the matrix of coefficients is not the identity. Let us stress that the function v need not be a solution to a Cauchy problem. Lemma 4.5.2 is an elementary statement which is needed in the proof of Lemma 4.3.4.

Lemma 4.5.1. *Set* $\Delta_a v := \sum_{k=1}^d \sum_j (\nabla_k + \beta_k^\mu)(a_{kj}\nabla_j v)$, $d \geq K$. *Then*

$$T_{0,a} = \| \, |\nabla v|_a \|_{2p}^{2p} \leq 2\|v\|_\infty^2 \| \, |\nabla v|_a^{p-2}\Delta_a v\|_2^2 + 2(p-1)^2\|v\|_\infty^2 J_{0,a},$$

for $p \geq 2$ ($J_{0,a} := \lim_{\varepsilon\to 0} J_{\varepsilon,a} = 4(p-1)^{-2}\| \, |\nabla|\nabla u|_a^{p-1}|_a\|_2^2$), *and*

$$T_{\varepsilon,a} \leq \varepsilon^{2p-2}\|v\|_2^2 + 2(\|v\|_\infty^2 + \varepsilon^2)\|\chi_\varepsilon \Delta_a v\|_2^2 + 2(p-1)^2\|v\|_\infty^2 J_{\varepsilon,a},$$

for $1 \leq p < 2$.

Lemma 4.5.2. *If* $3 - \frac{3}{1+\sqrt{1+3\varepsilon_0}} < p < \frac{2}{\varepsilon_0}$ *then there exist positive constants* $K(\varepsilon_0, p)$ *and* $C_{\varepsilon_0,p}$ *such that for sufficiently small* $\delta > 0$ *we have*

$$\begin{aligned}(1-\varepsilon_0)I_{\varepsilon,a} &+ \frac{2(p-2)-\varepsilon_0(p-2)^2}{4}J_{\varepsilon,a} - \varepsilon_0|p-2|\sqrt{I_{\varepsilon,a}J_{\varepsilon,a}} - \delta I_{\varepsilon,a} \\ &- c\delta(p-2)^2 J_{\varepsilon,a} \geq K(\varepsilon_0,p)J_{\varepsilon,a} - C_{\varepsilon_0,p}\sup_d \|\chi_\varepsilon|\nabla v|_a v_d\|_2^2. \end{aligned} \tag{4.38}$$

Moreover, if $p < 2$ *then*

$$\begin{aligned}(1-\varepsilon_0)I_{\varepsilon,a} &+ \frac{2(p-2)-\varepsilon_0(p-2)^2}{4}J_{\varepsilon,a} - \varepsilon_0|p-2|\sqrt{I_{\varepsilon,a}J_{\varepsilon,a}} - \delta I_{\varepsilon,a} \\ &- c\delta(p-2)^2 J_{\varepsilon,a} \geq \hat{K}(\varepsilon_0,p)I_{\varepsilon,a} - C_{\varepsilon_0,p}\sup_d \|\chi_\varepsilon|\nabla v|_a v_d\|_2^2. \end{aligned}$$

Proof of Lemma 4.5.1. Integration by parts yields

$$T_{\varepsilon,a} = \langle v[\nabla v]_{\varepsilon,a}^{2p-2}, \Delta_a v\rangle - (p-1)\langle v[\nabla v]_{\varepsilon,a}^{2p-4}(\nabla[\nabla v]_{\varepsilon,a}^2, \nabla v)_a\rangle. \tag{4.39}$$

Note that $\nabla[\nabla v]_{\varepsilon,a}^2 = \nabla|\nabla v|_a^2$. Therefore, $|(\nabla[\nabla v]_{\varepsilon,a}^2, \nabla v)_a| \leq |\nabla v|_a |\nabla|\nabla v|_a^2|_a$ by the Schwarz inequality. Thus, the absolute value of the last term in the right-hand side of (4.39) does not exceed

$$(p-1)\|v\|_\infty\langle [\nabla v]_{\varepsilon,a}^{2p-4}|\nabla v|_a|\nabla|\nabla v|_a^2|_a\rangle \leq \frac{1}{4}T_{\varepsilon,a} + (p-1)^2\|v\|_\infty^2 J_{\varepsilon,a}. \tag{4.40}$$

Consider the first term in the right-hand side of (4.39). Using the Young inequality we estimate it by

$$\begin{aligned}&\langle \chi_\varepsilon \varepsilon |v|,\, \chi_\varepsilon \varepsilon |\Delta_a v|\rangle + \langle \chi_\varepsilon |\nabla v|_a^2,\, \chi_\varepsilon |\Delta_a v||v|\rangle \\ &\leq \frac{1}{4}\|\chi_\varepsilon |\nabla v|_a^2\|_2^2 + \frac{\varepsilon^2}{4}\|\chi_\varepsilon v\|_2^2 + (\|v\|_\infty^2 + \varepsilon^2)\|\chi_\varepsilon \Delta_a v\|_2^2. \end{aligned} \tag{4.41}$$

If $p \geq 2$ then we can pass to the limit as $\varepsilon \to 0$ in (4.41). This yields the first assertion. If $p < 2$, then we observe that $\varepsilon^2\|\chi_\varepsilon v\|_2^2 \leq \varepsilon^{2p-2}\|v\|_2^2$ and $|\nabla v|_a^2 \chi_\varepsilon \leq [\nabla v]_{\varepsilon,a}^{p-1}|\nabla v|_a$ and combine (4.39)-(4.41) in order to obtain the second statement. □

Proof of Lemma 4.5.2. Let first $p \geq 2$. Setting $r := \left(\frac{I_{\varepsilon,a}}{J_{\varepsilon,a}}\right)^{\frac{1}{2}}$ we rewrite the left-hand side of (4.38) as follows

$$\frac{J_{\varepsilon,a}}{4}\left(4(1-\varepsilon_0-\delta)r^2 - 4\varepsilon_0|p-2|r + 2(p-2) - (\varepsilon_0 + 4c\delta)(p-2)^2\right) = \frac{J_{\varepsilon,a}}{4} F(r).$$

We need to find all p such that $F(r) > 0$, $r \geq 0$. A direct computation shows that if $p \in \left[2, 2/\varepsilon_0\right)$, then the discriminant of the quadratic function F is negative, provided δ is small enough.

Now we assume that $p < 2$. The following inequality holds.

$$J_{\varepsilon,a} \leq (4+\delta_1)I_{\varepsilon,a} + C_{\delta_1}\|\chi_\varepsilon |\nabla v|_a \Upsilon\|_2^2, \quad \forall \delta_1 > 0. \tag{4.42}$$

We give the proof of (4.42) below.

Making use of the Cauchy inequality and (4.42) we estimate the left-hand side of (4.38) from below by

$$\begin{aligned}&\Big[2(p-2) + 1 - \varepsilon_0 + 2\varepsilon_0(p-2) - \varepsilon_0(p-2)^2 - C_{\varepsilon_0,p}\delta\Big] I_{\varepsilon,a} - C_\delta \|\chi_\varepsilon |\nabla v|_a \Upsilon\|_2^2 \\ &= [G(p) - C_{\varepsilon_0,p}\delta] I_{\varepsilon,a} - C_\delta \|\chi_\varepsilon |\nabla v|_a \Upsilon\|_2^2,\end{aligned}$$

for every $\delta > 0$. It is easy to verify that if $p \in (3 - \frac{3}{1+\sqrt{1+3\varepsilon_0}}, 2)$, then $G(p) > 0$. Hence, $\hat{K}(\varepsilon_0, p) := G(p) - C_{\varepsilon_0,p}\delta > 0$ provided δ is small enough. This proves the second assertion. Inequality (4.38) now follows from (4.42).

Now we prove inequality (4.42). First we notice that for any $f, g, h \in \mathbb{R}^d$ the following inequality holds

$$\sum_{i,k,j,l=1}^{d} f_i\, a_{il}(\nabla_l a_{jk}) g_j h_k \leq v_d |f|_a |g|_a |h|_a. \tag{4.43}$$

We observe that

$$\nabla_k |\nabla v|_a^2 = \sum_{j,l\geq 1} 2(\nabla_{kj}^2 v) a_{jl} \nabla_l v + (\nabla_k a_{jl}) \nabla_j v \nabla_l v. \tag{4.44}$$

Hence,

$$|\nabla|\nabla v|_a^2|_a^2 \leq 2|AD^2 v|_{HS(a)}^2 |\nabla|\nabla v|_a^2|_a |\nabla v|_a + v_d |\nabla|\nabla v|_a^2|_a |\nabla v|_a^2.$$

This yields (4.42). □

Finally, we prove inequality (4.24). Since $\nabla[\nabla v]_{\varepsilon,a}^2 = \nabla|\nabla v|_a^2$, one gets

$$\begin{aligned}
&\sum_{i,j,k,l} a_{kj} \nabla_k v \nabla_j (a_{li} \nabla_i v) \nabla_l [\nabla v]_{\varepsilon,a}^{2p-4} \\
=&(p-2)[\nabla v]_{\varepsilon,a}^{2p-6} \sum_{i,j,k,l} \left(a_{kj} \nabla_k v (\nabla_j a_{li}) \nabla_i v \nabla_l |\nabla v|_a^2 + a_{li} a_{kj} \nabla_k v (\nabla_{ji}^2 v) \nabla_l |\nabla v|_a^2 \right).
\end{aligned}$$

Using (4.44) we obtain

$$\sum_{i,j,k,l} a_{li} a_{kj} \nabla_k v \nabla_{ji}^2 v \nabla_l |\nabla v|_a^2 = \frac{1}{2} |\nabla|\nabla v|_a^2|_a^2 - \frac{1}{2} \sum_{i,j,k,l} a_{kj} (\nabla_k |\nabla v|_a^2)(\nabla_j a_{li}) \nabla_i v \nabla_l v.$$

Recall that $J_{\varepsilon,a} = 4\langle [\nabla v]_{\varepsilon,a}^{2p-6} |\nabla v|_a^2 |\nabla|\nabla v|_a|_a^2 \rangle$. In order to estimate the remaining terms we employ (4.43):

$$\left| \sum_{i,j,k,l} [a_{kj} \nabla_k v (\nabla_j a_{li}) \nabla_i v \nabla_l |\nabla v|_a^2 \right| \leq v_d |\nabla|\nabla v|_a^2|_a |\nabla v|_a^2 \leq v_d |\nabla v|_a |\nabla|\nabla v|_a^2|_a [\nabla v]_{\varepsilon,a}.$$

The last term is estimated in the same manner. This yields (4.24).

Chapter 5

Global Gaussian Bounds with Applications to Semi-linear Problems

In this chapter we are concerned with some aspects of the qualitative theory of parabolic and elliptic equations, related to the differential expression

$$\Lambda := \nabla \cdot a(x) \cdot \nabla - b(x) \cdot \nabla - V(x), \ x \in \mathbb{R}^d, \ d \geq 3,$$

where the coefficients $a = (a_{ij})_{i,j=1}^d$, $b = (b_j)_{j=1}^d$ and V are measurable. Recall that we use the notation

$$\nabla \cdot a \cdot \nabla = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial}{\partial x_j}\Big), \quad b \cdot \nabla = \sum_{j=1}^{d} b_j \frac{\partial}{\partial x_j}.$$

Some essential background material on the subject is collected in section 2.6.

First we study the parabolic equation

$$\partial_t u(t,x) = \Lambda u(t,x) \tag{5.1}$$

in the domain $[0, \infty) \times \mathbb{R}^d$ and reveal sufficient conditions on the coefficients of Λ which ensure that the fundamental solution of (5.1) exists, is unique and enjoys global Gaussian bounds (see subsection 2.6.1 for relevant definitions). These become crucial when investigating the problem of existence/non-existence of positive weak solutions to semi-linear inequality

$$\Lambda u + u^p \leq 0, \ p > 1$$

in exterior domains (see subsection 2.6.4 for relevant definitions).

The method, employed in the investigation of non-linear elliptic equations, was proposed by V. Kondratiev.

5.1 Global Gaussian Bounds on Heat Kernels

In this section we study equation (5.1) and prove that under certain conditions, specified below, it has a unique heat kernel which satisfies global Gaussian upper and lower bounds.

5.1.1 Conditions on Coefficients and Formulation of Main Result

First we state the assumptions on the leading coefficients of equation (5.1). We assume that the matrix a is symmetric and satisfies the following conditions:

(A1) there is a constant $0 < \zeta < \infty$, such that, for all $x, z \in \mathbb{R}^d$ the inequalities

$$\zeta^{-1} \sum_{j=1}^{d} z_j^2 \leq \sum_{i,j=1}^{d} a_{ij}(x) z_i z_j \leq \zeta \sum_{j=1}^{d} z_j^2$$

hold;

(A2) for all $1 \leq i, j \leq d$ the functions a_{ij} are uniformly Hölder continuous.

Let $p = p(t, x, y)$ stand for the fundamental solution of the unperturbed equation

$$\partial_t u(t, x) = \nabla \cdot a(x) \cdot \nabla u(t, x).$$

Recall that Γ stands for the standard heat kernel, i.e. $\Gamma(t, x) := (4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}}$ for $t > 0$ and $x \in \mathbb{R}^d$ and $\Gamma_\alpha(t, x) := \Gamma(\alpha t, x)$ for $\alpha > 0$.

As was mentioned in subsection 2.6.1 conditions **(A1)** and **(A2)** imply that there exist positive constants β, c_β, $\bar{\beta}$, $c_{\bar{\beta}}$ such that

$$c_{\bar{\beta}} \Gamma_{\bar{\beta}}(t, x - y) \leq p(t, x, y) \leq c_\beta \Gamma_\beta(t, x - y) \tag{5.2}$$

and

$$|\nabla_x p(t, x, y)| \leq c_\beta t^{-\frac{1}{2}} \Gamma_\beta(t, x - y) \tag{5.3}$$

for all $t > 0$ and $x, y \in \mathbb{R}^d$.

We recall that if the coefficients $|b|$ and V belong to the enlarged Kato classes $\widehat{K}_{d+1}$ and $\widehat{K}_d$, respectively, then we can associate a C_0-semigroup $\exp(tH)$, $t \geq 0$, on $L^1(\mathbb{R}^d)$ with the differential expression Λ (for details see subsection 2.6.2).

In order to ensure the validity of global Gaussian bounds on the heat kernel of the operator H we have to impose more restrictive conditions on the lower order terms of Λ. Namely, we introduce the classes $\widehat{K}_{m,\infty}$, $m = d, d+1$, of *Green bounded potentials*.

Definition 5.1.1. *A potential $W \in L^1_{loc}(\mathbb{R}^d)$ is said to be in $\widehat{K}_{d,\infty}$ if*

$$M_d(W) := \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|W(y)|}{|x-y|^{d-2}} dy < \infty.$$

A potential W is said to belong to $\widehat{K}_{d+1,\infty}$ if

$$M_{d+1}(W) := \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|W(y)|}{|x-y|^{d-1}} dy < \infty.$$

It is well-known (see e.g., [71]) that if $W \in \widehat{K}_{d,\infty}$ and $M_d(W)$ is sufficiently small, then (2.26) holds with some $\delta \in (0,1)$ and $c(\delta) = 0$.

Throughout the first part of this section we assume that $|b| \in \widehat{K}_{d+1,\infty}$. Another condition, frequently imposed on the drift coefficient b, is $|b|^2 \in \widehat{K}_{d,\infty}$. Two examples below show the relation between these conditions.

First we note that a potential $W = c|x|^\alpha \wedge 1$ belongs to $\widehat{K}_{d,\infty}$ iff $\alpha < -2$. Indeed,

$$\int_{\mathbb{R}^d} \frac{|W(y)|}{|x-y|^{d-2}} dy \leq C + c\int_{B_R^c} \frac{|y|^\alpha}{|x-y|^{d-2}} dy \leq C + c\int_{B_R^c} |y|^{\alpha+2-d} \leq C_1,$$

provided $\alpha < -2$ for all $x \in \mathbb{R}^d$, so $M_d(W)$ is finite. Similarly, the potential $W \in \widehat{K}_{d+1,\infty}$ iff $\alpha < -1$. Hence, for drift coefficients of the form $|b(x)| = c|x|^\alpha \wedge 1$, $x \in \mathbb{R}^d$, we have

$$|b| \in \widehat{K}_{d+1,\infty} \iff |b|^2 \in \widehat{K}_{d,\infty},$$

namely when $\alpha < -1$.

Next we assume that $W(x) = c|x|^\alpha \log^\beta |x| \wedge 1$, $x \in \mathbb{R}^d$. One can check that the potentials $W_1 = c|x|^{-2}\log^\beta |x| \wedge 1$ and $W_2 \in c|x|^{-1}\log^\beta |x| \wedge 1$ belong to $\widehat{K}_{d,\infty}$ and $\widehat{K}_{d+1,\infty}$, respectively, iff $\beta < -1$. Suppose $|b(x)| \leq c|x|^{-1}\log^\beta |x|$. A direct computation shows that $|b| \in \widehat{K}_{d+1,\infty}$ iff $\beta < -1$, and $|b|^2 \in \widehat{K}_{d,\infty}$ iff $\beta < -1/2$. The last example demonstrates the difference between the two conditions.

Next we state the assumptions on the lower order terms of equation (5.1).

(B1) We assume that $|b| \in \widehat{K}_{d+1,\infty}$ and the number $M_{d+1}(b)$ is sufficiently small.

(C1) We assume that $V \in \widehat{K}_{d,\infty}$ and the number $M_d(V)$ is sufficiently small.

The main result of this section reads as follows.

Theorem 5.1.2. *Let conditions* **(A1)-(A2)**, **(B1)** *and* **(C1)** *hold. Then the weak fundamental solution $r = r(t,x,y)$ of equation* (5.1) *exists, is unique and satisfies the estimate*

$$C_{\bar{\gamma}}\Gamma_{\bar{\gamma}}(t, x-y) \le r(t,x,y) \le C_\gamma \Gamma_\gamma(t,x-y),\ t > 0,\ x, y \in \mathbb{R}^N,$$

with some positive constants $\gamma, C_\gamma, \bar{\gamma}, C_{\bar{\gamma}}$.

In the study of semi-linear inequalities in exterior domains (see Theorem 5.2.1 below) we need global Gaussian estimates for the fundamental solution of (5.1). For that reason we require that the lower order coefficients satisfy global conditions **(B1)** and **(C1)**. However, if we restrict ourselves to the problem of existence of heat kernels and validity of local Gaussian bounds, then the conditions on b and V can be relaxed. Namely, we can assume that $b \in \widehat{K}_{d+1}$, $V \in \widehat{K}_d$ and $M_{d+1}(b,\rho)$, $M_d(V,\rho)$ are sufficiently small for some $\rho > 0$. The proofs in subsection 5.1.3 remain the same, with $M_{d+1}(b)$ and $M_d(V)$ replaced by $M_{d+1}(b,\rho)$ and $M_d(V,\rho)$, respectively.

Our strategy to prove Theorem 5.1.2 is as follows. First we derive a priori Gaussian estimates of fundamental solutions of equations with bounded coefficients. It is vital, however, that these estimates are independent of L^∞-norms of b and V. We then use a semigroup approach to prove that the semigroup $\exp(tH)$ is integral and the corresponding integral kernel r enjoys two-sided Gaussian bounds. Finally we obtain additional bounds on the gradients of weak solutions to auxiliary Cauchy problems in order to show, by a limiting argument, that r is a fundamental solution of (5.1). Uniqueness of the heat kernel is established by standard means.

The following result for the corresponding elliptic equation is a direct consequence of Theorem 5.1.2.

Corollary 5.1.3. *Let $G = G(x, y)$ stand for the fundamental solution of the equation $(\Lambda v)(x) = 0$. We assume that the conditions of Theorem 5.1.2 are fulfilled. Then there exist constants $C_1, C_2 > 0$ such that*

$$C_1|x-y|^{2-d} \le G(x,y) \le C_2|x-y|^{2-d} \quad \textit{for} \quad x, y \in \mathbb{R}^d,\ x \ne y.$$

Proof. The proof immediately follows from Theorem 5.1.2 and the well-known equality

$$G(x,y) = \int_0^\infty r(t,x,y)\,dt.$$

□

5.1.2 Estimates of Integral Kernels Corresponding to First Order Perturbations

The proof of a priori Gaussian bounds for equation (5.1) is based on the following lemma.

Lemma 5.1.4. *Let $0 < \delta_2 < \delta_1$. Then there exist a constant $C_{\delta_1,\delta_2} > 0$ such that*

(i) $\int_0^t \int_{\mathbb{R}^d} \Gamma_{\delta_1}(t-s, x-z)|b(z)|\Gamma_{\delta_2}(s, z-y)s^{-\frac{1}{2}}dzds$
$\le C_{\delta_1,\delta_2} M_{d+1}(b)\Gamma_{\delta_1}(t, x-y),$

(ii) $\int_0^t \int_{\mathbb{R}^d} (t-s)^{-\frac{1}{2}}\Gamma_{\delta_1}(t-s, x-z)|b(z)|\Gamma_{\delta_2}(s, z-y)s^{-\frac{1}{2}}dzds$
$\le C_{\delta_1,\delta_2} M_{d+1}(b)t^{-\frac{1}{2}}\Gamma_{\delta_1}(t, x-y),$

(iii) $\int_0^t \int_{\mathbb{R}^d} \Gamma_{\delta_1}(t-s, x-z)|V(z)|\Gamma_{\delta_2}(s, z-y)dzds$
$\le C_{\delta_1,\delta_2} M_d(V)\Gamma_{\delta_1}(t, x-y),$

for all $t > 0$ and $x, y \in \mathbb{R}^d$.

Remark. *This lemma was proved in [81, Lemma 3.1]. However, because of its importance for the further study and in order to keep the presentation self-contained, we give a complete proof of statement (i).*

Proof. We use the notation

$$I := \left(4\pi\sqrt{\delta_1\delta_2}\right)^d \int_0^t \int_{\mathbb{R}^d} \Gamma_{\delta_1}(t-s, x-z)|b(z)|\Gamma_{\delta_2}(s, z-y)s^{-\frac{1}{2}}dzds.$$

We also set $a_i := \dfrac{1}{4\delta_i}$, $i = 1, 2$, and note that $0 < a_1 < a_2$.

Let $\varepsilon \in (0, 1)$. Then

$$I = \left(4\pi\sqrt{\delta_1\delta_2}\right)^d \left(\int_0^{\varepsilon t} \int_{\mathbb{R}^d} \Gamma_{\delta_1} |b| \Gamma_{\delta_2} s^{-\frac{1}{2}} dz ds + \int_{\varepsilon t}^{t} \int_{\mathbb{R}^d} \Gamma_{\delta_1} |b| \Gamma_{\delta_2} s^{-\frac{1}{2}} dz ds \right) =: I_1 + I_2.$$

First we estimate I_1. A direct computation yields

$$\begin{aligned} I_1 &= \int_0^{\varepsilon t} (t-s)^{-d/2} s^{-d/2} \int_{\mathbb{R}^d} e^{-a_1 \frac{|x-z|^2}{t-s}} |b(z)| e^{-a_2 \frac{|y-z|^2}{s}} dz ds \\ &= \int_0^{\varepsilon t} (t-s)^{-d/2} s^{-(d+1)/2} \int_{\mathbb{R}^d} e^{-a_1 \left(\frac{|x-z|^2}{t-s} + \frac{|y-z|^2}{s} \right)} |b(z)| e^{-(a_2 - a_1) \frac{|y-z|^2}{s}} dz ds \end{aligned}$$

Next we observe that

$$\frac{|x-z|^2}{t-s} + \frac{|y-z|^2}{s} \geq \frac{|x-y|^2}{t} \quad \text{for all } x, y, z \in \mathbb{R}^d \text{ and } 0 < s < t.$$

Therefore

$$e^{-a_1 \left(\frac{|x-z|^2}{t-s} + \frac{|y-z|^2}{s} \right)} \leq e^{-a_1 \frac{|x-y|^2}{t}}.$$

Furthermore, $t - s \geq (1 - \varepsilon)t$ for all $s \in (0, \varepsilon t)$. Hence, we obtain that

$$I_1 \leq \left((1-\varepsilon)t\right)^{-d/2} e^{-a_1 \frac{|x-y|^2}{t}} \int_0^{\varepsilon t} s^{-(d+1)/2} \int_{\mathbb{R}^d} |b(z)| e^{-(a_2 - a_1) \frac{|y-z|^2}{s}} dz ds. \tag{5.4}$$

Now we turn to estimating I_2. Set $\delta := (a_1/a_2)^{1/2}$. We have

$$\begin{aligned} I_2 &= \int_{\varepsilon t}^{t} (t-s)^{-d/2} s^{-d/2} \int_{\mathbb{R}^d} e^{-a_1 \frac{|x-z|^2}{(t-s)}} |b(z)| e^{-a_2 \frac{|y-z|^2}{s}} dz ds \\ &= \int_{\varepsilon t}^{t} \left(\int_{|z-y| \geq |x-y|\delta} + \int_{|z-y| \leq |x-y|\delta} \right) \ldots dz ds = I_{21} + I_{22} \end{aligned}$$

When $|z - y| \geq |x - y|\delta$ and $s \in (\varepsilon t, t)$ we get

$$s^{-d/2} e^{-a_2 \frac{|z-y|^2}{s}} \leq (\varepsilon t)^{-d/2} e^{-a_1 \frac{|x-y|^2}{t}} \quad \text{and} \quad s^{-1} \leq \varepsilon^{-1}(1-\varepsilon)(t-s)^{-1}.$$

Therefore

$$\begin{aligned} I_{21} &\leq \frac{e^{-a_1 \frac{|x-y|^2}{t}}}{(\varepsilon t)^{d/2}} \int_{\varepsilon t}^{t} (t-s)^{-d/2} s^{-1/2} \int_{|z-y| \geq |x-y|\delta} |b(z)| e^{-a_1 \frac{|x-z|^2}{t-s}} dz ds \\ &\leq \sqrt{\frac{1-\varepsilon}{\varepsilon}} \frac{e^{-a_1 \frac{|x-y|^2}{t}}}{(\varepsilon t)^{d/2}} \int_{\varepsilon t}^{t} \int_{|z-y| \geq |x-y|\delta} \frac{|b(z)| e^{-a_1 \frac{|x-z|^2}{t-s}}}{(t-s)^{(d+1)/2}} dz ds \end{aligned} \tag{5.5}$$

Using a similar argument we get

$$I_{22} \le (\varepsilon t)^{-d/2} \int_{\varepsilon t}^{t} \int_{|z-y|\le|x-y|\delta} \frac{|b(z)|e^{-a_1\frac{|x-z|^2}{t-s}}}{s^{1/2}(t-s)^{d/2}} dzds$$

$$\le \sqrt{\frac{1-\varepsilon}{\varepsilon}} (\varepsilon t)^{-d/2} \int_{\varepsilon t}^{t} \int_{|z-y|\le|x-y|\delta} \frac{|b(z)|e^{-a_1\frac{|x-z|^2}{t-s}}}{(t-s)^{(d+1)/2}} dzds.$$

Since $|z-y| \le |x-y|\delta$ we have

$$|x-z| \ge |x-y| - |y-z| \ge |x-y|(1-\delta).$$

Therefore

$$e^{-a_1\frac{|x-z|^2}{t-s}} = e^{-a_1\frac{|x-z|^2}{2(t-s)}} e^{-a_1\frac{|x-z|^2}{2(t-s)}} \le e^{-a_1\frac{|x-z|^2}{2(t-s)}} e^{-a_1\frac{|x-y|^2}{2(t-s)}(1-\delta)}$$

$$\le e^{-a_1\frac{|x-z|^2}{2(t-s)}} e^{-a_1\frac{|x-y|^2}{2(1-\varepsilon)t}(1-\delta)},$$

since $t-s < (1-\varepsilon)t$. Now choosing ε in such way that

$$\frac{1-\delta}{2(1-\varepsilon)} = 1,$$

we derive the estimate

$$e^{-a_1\frac{|x-z|^2}{t-s}} \le e^{-a_1\frac{|x-z|^2}{2(t-s)}} e^{-a_1\frac{|x-y|^2}{t}}.$$

The last inequality implies that

$$I_{22} \le \sqrt{\frac{1-\varepsilon}{\varepsilon}} \frac{e^{-a_1\frac{|x-y|^2}{t}}}{(\varepsilon t)^{d/2}} \int_{\varepsilon t}^{t} \int_{|z-y|\le|x-y|\delta} \frac{|b(z)|e^{-a_1\frac{|x-z|^2}{2(t-s)}}}{(t-s)^{(d+1)/2}} dzds \tag{5.6}$$

We combine estimates (5.4), (5.5) and (5.6) and observe that for all $t > 0$ and $a > 0$ there is a constant $C = C(a) > 0$ independent of t such that

$$\int_0^t \int_{\mathbb{R}^d} |b(z)|e^{-a\frac{|x-z|^2}{s}} s^{-(d+1)/2} dzds \le CM_{d+1}(b).$$

This completes the proof of (i). The proofs of (ii)-(iv) are similar and therefore omitted. □

Employing Lemma 5.1.4 one can derive the Gaussian bounds on the heat kernel of the equation

$$\partial_t u(t,x) = \nabla \cdot a(x) \cdot \nabla u(t,x) - b(x) \cdot \nabla u(t,x)$$

with $|b| \in L^\infty(\mathbb{R}^N)$ (the existence of the heat kernel follows from [7]). The following statement is due to Q. Zhang (see [81, Theorem A, Corollary 1.1]).

Proposition 5.1.5. *Suppose that conditions* **(A1)**, **(A2)** *and* **(B1)** *are fulfilled and the drift b is bounded. Then for every $\alpha > \beta$ there are positive constants $\bar{\alpha}$, C_α, $C_{\bar{\alpha}}$ such that*

(i) $C_{\bar{\alpha}}\Gamma_{\bar{\alpha}}(t,\, x-y) \le q(t,x,y) \le C_\alpha\Gamma_\alpha(t,\, x-y)$,

(ii) $|\nabla_x q(t,x,y)| \le C_\alpha t^{-1/2}\Gamma_\alpha(t,\, x-y)$,

for all $t > 0$ and $x, y \in \mathbb{R}^d$.

Proof. (i). It follows from the Duhamel principle that

$$q(t,x,y) = p(t,x,y) - \int_0^t \int_{\mathbb{R}^d} q(t-s,x,z)b(z)\cdot\nabla_z p(s,z,y)dzds.$$

This implies that q can be represented formally as

$$q(t,x,y) = \sum_{n=0}^{\infty} J_n(t,x,y), \tag{5.7}$$

where $J_0(t,x,y) = p(t,x,y)$ and

$$J_n(t,x,y) = -\int_0^t \int_{\mathbb{R}^d} J_{n-1}(t-s,x,z)b(z)\cdot\nabla_z p(s,z,y)dzds,\ n \in \mathbb{N}.$$

It follows from (5.2) that

$$J_0(t,x,y) \le C_\beta\Gamma_\alpha(t,\, x-y).$$

Estimate (5.3) and Lemma 5.1.4, (i) imply that

$$\begin{aligned} J_1(t,x,y) &\le C_\beta^2 \int_0^t \int_{\mathbb{R}^d} \Gamma_\alpha(t-s,\, x-z)|b(z)|s^{-1/2}\Gamma_\beta(s,\, z-y)dzds \\ &\le C_\beta^2 C_{\alpha,\beta} M_{d+1}(b)\Gamma_\alpha(t, x-y). \end{aligned}$$

The principle of mathematical induction yields

$$|J_n(t,x,y)| \le C_\beta(C_\beta C_{\alpha,\beta}M_{d+1}(b))^n\Gamma_\alpha(t,x-y),\ n \in \mathbb{N}.$$

If $C_\beta C_{\alpha,\beta}M_{d+1}(b) < 1$ then the series (5.7) converges and

$$q(t,x,y) \le \frac{C_\beta}{1 - C_\beta C_{\alpha,\beta}M_{d+1}(b)}\Gamma_\alpha(t,x-y).$$

Next, making use of (5.2) we conclude that

$$\begin{aligned} q(t,x,y) &\geq p(t,x,y) - \sum_{n=1}^{\infty} |J_n(t,x,y)| \\ &\geq C_\beta \Gamma_\beta(t,x-y) - \frac{C_\beta^2 C_{\alpha,\beta} M_{d+1}(b)}{1 - C_\beta C_{\alpha,\beta} M_{d+1}(b)} \Gamma_\alpha(t,x-y). \end{aligned}$$

This implies the existence of a constant C_0 such that

$$q(t,x,y) \geq C_0 t^{-d/2},$$

provided $|x-y|^2 < t$. Now Gaussian lower bound for q follows from the last estimate by a standard argument (see e.g. [26]).

(ii). Formal differentiation of equality (5.7) gives

$$\nabla_x q(t,x,y) = \sum_{n=0}^{\infty} \nabla_x J_n(t,x,y).$$

By (5.3)

$$|\nabla_x J_0(t,x,y)| \leq C_\beta t^{-1/2} \Gamma_\alpha(t,x-y).$$

In order to estimate $|\nabla_x J_1(t,x,y)|$ we employ Lemma 5.1.4,(ii).

$$\begin{aligned} |\nabla_x J_1(t,x,y)| &\leq C_\beta^2 \int_0^t \int_{\mathbb{R}^d} (t-s)^{-1/2} \Gamma_\alpha(t-s,\, x-z) |b(z)| s^{-1/2} \Gamma_\beta(s,\, z-y) dz ds \\ &\leq C_\beta^2 C_{\alpha,\beta} M_{d+1}(b) t^{-1/2} \Gamma_\alpha(t,x-y). \end{aligned}$$

The rest of the proof is the same as the that of the upper bound in part (i). □

Now let $b \in \widehat{K}_{d+1,\infty}$. We set $b^k := (b_1^k, \ldots, b_d^k)$, $k \in \mathbb{N}$, where $b_j^k := b_j \mathbb{1}_{|b| \leq k}$. One can readily see that

$$b^k \in \widehat{K}_{d+1,\infty} \cap \left(L^\infty(\mathbb{R}^d)\right)^d \text{ and } M_{d+1}(b^k) \leq M_{d+1}(b).$$

By $q_k = q_k(t,x,y)$ we denote the fundamental solution of the equation

$$\partial_t u(t,x) = \nabla \cdot a(x) \cdot \nabla u(t,x) - b^k(x) \cdot \nabla u(t,x). \tag{5.8}$$

It follows from Proposition 5.1.5 that there exist positive constants α, $\tilde{\alpha}$, C_α, $C_{\tilde{\alpha}}$, independent of k, such that

$$\begin{aligned} C_{\tilde{\alpha}} \Gamma_{\tilde{\alpha}}(t,\, x-y) \leq q_k(t,x,y) &\leq C_\alpha \Gamma_\alpha(t,\, x-y), \\ |\nabla_x q_k(t,x,y)| &\leq C_\alpha t^{-1/2} \Gamma_\alpha(t,\, x-y). \end{aligned} \tag{5.9}$$

Let $\mathcal{L}_k = \mathcal{A} + \mathcal{B}_k$, $\mathcal{D}(\mathcal{L}_k) = \mathcal{D}(\mathcal{A})$, where the operator $\mathcal{B}_k$ is defined in the same way as $\mathcal{B}$, with b replaced by b^k. By $\mathcal{L}$ we denote the operator H with $V = 0$. Let $f \in \mathcal{D}(\mathcal{A})$. By the Duhamel principle we have

$$\exp(t\mathcal{L}_k)f - \exp(t\mathcal{L})f = \int_0^t \exp\left((t-s)\mathcal{L}\right)\left(\mathcal{L} - \mathcal{L}_k\right)\exp\left(s\mathcal{L}_k\right) f ds.$$

Therefore, for a fixed $t > 0$, we get

$$\begin{aligned}
\|\exp(t\mathcal{L}_k)f - \exp(t\mathcal{L})f\|_1 &\leq \int_0^t \|\exp\left((t-s)\mathcal{L}\right)\left(\mathcal{L} - \mathcal{L}_k\right)\exp\left(s\mathcal{L}_k\right) f\|_1 ds \\
&\leq C \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\omega(t-s)}|b(x)|\mathbb{1}_{|b|>k}|\nabla_x q_k(s,x,y)|\,|f(y)|dxdyds \\
&\leq \hat{C} \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |b(x)|\mathbb{1}_{|b|>k} s^{-1/2}\Gamma_\alpha(s, x-y)|f(y)|dxdyds \\
&\leq \hat{C}\, M_{d+1}(|b|, t, \gamma)\|f\|_1, \qquad (5.10)
\end{aligned}$$

where the last inequality in (5.10) follows from (2.30) and the Fubini theorem. Since $|b|\mathbb{1}_{|b|>k} \to 0$ a.e., the Lebesgue dominated convergence theorem implies that

$$\exp(t\mathcal{L}_k)f \to \exp(t\mathcal{L})f \quad \text{in} \quad L^1(\mathbb{R}^d) \quad \text{for all} \quad f \in \mathcal{D}(\mathcal{A}).$$

Observing that $\mathcal{D}(\mathcal{A})$ is dense in $L^1(\mathbb{R}^d)$ we conclude that $\exp(t\mathcal{L}_k) \to \exp(t\mathcal{L})$ strongly in $L^1(\mathbb{R}^d)$.

Next we claim that $\|\exp(t\mathcal{L}) \restriction_{L^1\cap L^\infty} \|_{\infty\to\infty} \leq 1$. Indeed, let $f \in L^1 \cap L^\infty(\mathbb{R}^d)$. Set $h := \exp(t\mathcal{L})f$, $h_k := \exp(t\mathcal{L}_k)f$. Since $h_k \to h$ in $L^1(\mathbb{R}^d)$ one can find a subsequence h_{k_l} such that $h_{k_l} \to h$ a.e. We note that for every $k \in \mathbb{N}$ the semigroup $\exp(t\mathcal{L}_k)$ is sub-Markovian. Therefore

$$|\exp(t\mathcal{L}_k)f|(x) = \left|\int_{\mathbb{R}^d} q_k(t,x,y)f(y)dy\right| \leq \|f\|_\infty,$$

for all $k \in \mathbb{N}$.

The inequality $\|f\|_p^p \leq \|f\|_1\|f\|_\infty^{p-1}$ yields $\exp(t\mathcal{L}_k)f \to \exp(t\mathcal{L})f$ in $L^p(\mathbb{R}^d)$ for all $1 \leq p < \infty$ and $f \in L^1 \cap L^\infty(\mathbb{R}^d)$. Hence, making use of (2.25) we conclude that there is a constant $c > 0$ such that

$$\|\exp(t\mathcal{L})f\|_p = \lim_k \|\exp(t\mathcal{L}_k)f\|_p \leq ct^{-d/2p}\|f\|_1, \qquad (5.11)$$

for all $p > 1$, i.e. $\|\exp(t\mathcal{L})\|_{1\to p} \leq ct^{-d/2p}$. Hence, $\exp(t\mathcal{L})$ is an integral semigroup, i.e. there is a function $q = q(t,x,y)$ such that for all $f \in L^1(\mathbb{R}^d)$ we have

$$\big(\exp(t\mathcal{L})f\big)(x) = \int\limits_{\mathbb{R}^d} q(t,x,y)f(y)dy.$$

Next we observe that

$$\lim_k \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} q_k(t,x,y)f(y)g(x)dydx = \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} q(t,x,y)f(y)g(x)dydx.$$

for all $0 \leq f \in L^1(\mathbb{R}^d)$ and $0 \leq g \in L^\infty(\mathbb{R}^d)$. Using (5.9) we infer that

$$\begin{aligned}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} C_{\tilde{\alpha}}\Gamma_{\tilde{\alpha}}(t,\, x-y)f(y)g(x)dydx &\leq \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} q(t,x,y)f(y)g(x)dydx \\ &\leq \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} C_\alpha\Gamma_\alpha(t,\, x-y)f(y)g(x)dydx.\end{aligned}$$

For $z \in \mathbb{R}^d$ set $Q_{n,z} := \prod_{j=1}^{d}[z_j - 1/n,\, z_j + 1/n]$. For fixed $x^0, y^0 \in \mathbb{R}^d$ we take $f = f_n := 1\!\!1_{Q_{n,y^0}}$ and $g = g_n := 1\!\!1_{Q_{n,x^0}}$, $n \in \mathbb{N}$, in the last inequality and pass to the limit as $n \to \infty$. By the Lebesgue-Besicovitch differentiation theorem (see e.g. [22, 1.7, Theorem 1]) the kernel q satisfies the estimate

$$C_{\tilde{\alpha}}\Gamma_{\tilde{\alpha}}(t,\, x-y) \leq q(t,x,y) \leq C_\alpha\Gamma_\alpha(t,\, x-y), \tag{5.12}$$

for all $t > 0$ and a.e. $x, y \in \mathbb{R}^d$.

We complete this subsection by establishing an estimate of the gradient of q.

Lemma 5.1.6. *For every $t > 0$ and a.e. $y \in \mathbb{R}^d$ the vector field $\nabla_x q(t,\cdot,y) \in L^2(\mathbb{R}^d)$ and the estimate*

$$|\nabla_x q(t,x,y)| \leq C_\gamma\Gamma_\gamma(t, x-y) \tag{5.13}$$

holds for a.e. $t > 0$, $x, y \in \mathbb{R}^d$, where γ and C_γ are the same as in (5.9).

Proof. First we observe that

$$\int\limits_{\mathbb{R}^d} q_k(t,x,y)g(x)dx \to \int\limits_{\mathbb{R}^d} q(t,x,y)g(x)dx$$

for all $g \in L^2(\mathbb{R}^d)$ by (5.9) and an argument similar to the proof of eqreflpest.

The second estimate in (5.9) implies that

$$\int_{\mathbb{R}^d} |\nabla_x q_k(t,x,y)|^2 dx \le Ct^{-(d+1)} \int_{\mathbb{R}^d} \exp(-2\gamma|x-y|^2/t)dx \le Ct^{-\frac{d+2}{2}}.$$

The weak compactness of a ball in $L^2(\mathbb{R}^d)$ yields the existence of a subsequence $(q_{k_l})_{k\in\mathbb{N}}$ and a vector field $h \in L^2(\mathbb{R}^d)$ such that $\nabla_x r_{k_l} \to h$ weakly in $L^2(\mathbb{R}^d)$. Noting that the operator ∇ is closed w.r.t. weak convergence in $\left(L^2(\mathbb{R}^d)\right)^d$ we see that $h = \nabla_x q$ a.e. Therefore $\nabla_x q_k(t,\cdot,y) \to \nabla_x q(t,\cdot,y)$. Similar to the proof of (5.12) we use estimate (5.9), apply the Lebesgue-Besicovitch theorem and see that (5.13) holds. □

5.1.3 Existence and Uniqueness of Fundamental Solution

In this subsection we prove Theorem 5.1.2. First we establish several auxiliary results.

Let $|b| \in \widehat{K}_{d+1,\infty}$ and $V \in \widehat{K}_{d,\infty}$. We set $V^n := V\mathbb{1}_{|V|\le n}$. One can readily see that

$$V^n \in \widehat{K}_{d,\infty} \cap L^\infty(\mathbb{R}^d) \quad \text{and} \quad M_d(V^n) \le M_d(V).$$

Let $H_n := \mathcal{L} + \mathcal{V}_n$, $\mathcal{D}(H_n) = \mathcal{D}(\mathcal{A})$, $n \in \mathbb{N}$, where the operator $\mathcal{V}_n$ is defined in the same way as $\mathcal{V}$, with V replaced by V^n. It is clear that the semigroup $\exp(tH_n)$, $t \ge 0$, is integral for every $n \in \mathbb{N}$. By $r_n = r_n(t,x,y)$ we denote the corresponding kernel. The following statement holds.

Proposition 5.1.7. *Suppose that conditions* **(A1)**, **(A2)**, **(B1)** *and* **(C1)** *are fulfilled. Then for all $\gamma > \alpha$ there are positive constants $\bar{\gamma}$, C_γ, $C_{\bar{\gamma}}$ and $C_{\gamma,n} = C(\gamma, \|V^n\|_\infty)$ such that*

(i) $C_{\bar{\gamma}}\Gamma_{\bar{\gamma}}(t,\, x-y) \le r_n(t,x,y) \le C_\gamma\Gamma_\gamma(t,\, x-y)$,

(ii) $|\nabla_x r_n(t,x,y)| \le C_{\gamma,n} t^{-1/2}\Gamma_\gamma(t,\, x-y)$,

for all $t > 0$ and $x, y \in \mathbb{R}^d$.

Remark. *In the case $b = 0$ statement (i) was proved in [82].*

Proof. We only observe that by the Duhamel principle we have

$$r_n(t,x,y) = q(t,x,y) - \int_0^t \int_{\mathbb{R}^d} q(t-s,x,z)V^n(z)r_n(s,z,y)dzds.$$

The rest of the proof is similar to that of Proposition 5.1.5. One makes use of Lemma 5.1.4 (iii) in order to prove statement (i). Estimate (ii) can be derived by differentiating the above equality and employing the upper bound in (i) and Lemma 5.1.6. □

Repeating the corresponding arguments from the previous subsection and making use of Proposition 5.1.7(i) we conclude that $\exp(tH_n) \to \exp(tH)$ strongly in $L^p(\mathbb{R}^d)$, $1 \le p < \infty$. Similarly we infer that the semigroup $\exp(tH)$, $t \ge 0$, is integral and its kernel $r = r(t,x,y)$ enjoys the estimates

$$C_{\bar{\gamma}}\Gamma_{\bar{\gamma}}(t,\, x-y) \le r(t,x,y) \le C_\gamma \Gamma_\gamma(t,\, x-y), \tag{5.14}$$

for all $t > 0$ and $x, y \in \mathbb{R}^d$.

Our next goal is to prove that r is a weak fundamental solution of (5.1), i.e. that the function $u(t,x) := \int_{\mathbb{R}^d} r(t,x,y)f(y)dy$, $0 < t \le T$, $x \in \mathbb{R}^d$, is a weak solution to the Cauchy problem

$$\begin{aligned} &\partial_t u = \nabla \cdot a \cdot \nabla u - b \cdot \nabla u - Vu, \ 0 < t \le T, \\ &u(0) = f, \end{aligned} \tag{5.15}$$

for every $T > 0$ and $f \in L_c^\infty(\mathbb{R}^d)$ (we recall that $L_c^\infty(\mathbb{R}^d)$ stands for the space of compactly supported functions in $L^\infty(\mathbb{R}^d)$). The following proposition is an important step in proving Theorem 5.1.2.

Proposition 5.1.8. *Let $T > 0$. Let $u(t,x) := \int_{\mathbb{R}^d} r(t,x,y)f(y)dy$, $0 < t < T$, with $f \in L^1(\mathbb{R}^d)$. Then $|\nabla u| \in \left(P\widehat{K}_{d+1}^\gamma\right)^*$ and*

$$\| \, |\nabla u| \, \|_{(P\widehat{K}_{d+1}^\gamma)^*} \le C_\gamma\big(1 + M_d(V)\big)\|f\|_1,$$

where $\left(P\widehat{K}_{d+1}^\gamma\right)^*$ stands for the dual to $P\widehat{K}_{d+1}^\gamma$.

Proof. For $t \in (0,T)$, $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$ we set $u_n(t,x) := \int_{\mathbb{R}^d} r_n(t,x,y)f(y)dy$, where $f \in L^1(\mathbb{R}^d)$. It follows from Proposition 5.1.7(ii) that $|\nabla u_n| \in \left(P\widehat{K}_{d+1}^\gamma\right)^*$. Let $w \in P\widehat{K}_{d+1}^\gamma$. Making successive use of the Duhamel principle, Lemma 5.1.6, Proposition 5.1.7(i) and the Fubini theorem we get

$$\sup_y \int_0^T \int_{\mathbb{R}^d} |w(t,x)||\nabla_x r_n(t,x,y)|dxdt \le \sup_y \Bigg[\int_0^T \int_{\mathbb{R}^d} |w(t,x)||\nabla_x q(t,x,y)|dxdt$$
$$+ \int_0^T \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} |w(t,x)||\nabla_x q(t-s,x,z)||V^n(z)|r_n(s,z,y)dzdsdxdt\Bigg]$$
$$\le C_\gamma \sup_y \Bigg[\int_0^T \int_{\mathbb{R}^d} |w(t,x)|t^{-1/2}\Gamma_\gamma(t,x-y)dxdt$$
$$+ \int_0^T \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} |w(t,x)|(t-s)^{-1/2}\Gamma_\gamma(t-s,x-z)|V(z)|\Gamma_\gamma(s,z-y)dzdsdxdt\Bigg]$$
$$\le C_\gamma\big(1+M_d(V)\big)M_{d+1}(w,T,\gamma). \tag{5.16}$$

Hence, the sequence $(\nabla u_n)_{n\in\mathbb{N}}$ is compact in the w^*-topology of $\big(P\widehat{K}^\gamma_{d+1}\big)^*$ by the Alaoglu theorem (see [17, Th. V.4.2]). We note that $u_n \to u$ in $L^1(\mathbb{R}^d)$ locally uniformly in t. Next we observe that $L^\infty((0,T)\times\mathbb{R}^d) \subset P\widehat{K}^\gamma_{d+1}$. This implies that a sub-sequence of $(\nabla u_n)_{n\in\mathbb{N}}$ is weakly convergent in $\Big(L^1(\mathbb{R}^d)\Big)^d$. We make use of the closedness of the gradient and conclude that $\nabla u_n \to \nabla u$ in the w^*-topology of $(P\widehat{K}_{d+1,\gamma})^*$. Thus $|\nabla u| \in (P\widehat{K}_{d+1,\gamma})^*$. The stated estimate is a direct consequence of (5.16). □

By $r_{kn} = r_{kn}(t,x,y)$, $k,n \in \mathbb{N}$, we denote the fundamental solution of the equation

$$\partial_t u = \nabla\cdot a\cdot\nabla u - b^k\cdot\nabla u - V^n u$$

(existence of the heat kernel follows from [7]). It is clear that both statements of Proposition 5.1.7 are valid for r_{kn}. We set $u_{kn}(t,x) := \int_{\mathbb{R}^d} r_{kn}(t,x,y)f(y)$, $f \in L^\infty_c(\mathbb{R}^d)$. One can verify directly that u_{kn} is a solution to the problem

$$u_{kn} \in C([0,T];L^2(\mathbb{R}^d)) \cap L^2((0,T);H^1(\mathbb{R}^d)),$$
$$b^k\cdot\nabla u_{kn},\ V^n u_{kn} \in L^1\big((0,T)\times\mathbb{R}^d\big),$$
$$\int_0^T \int_{\mathbb{R}^d} (\nabla u_{kn}\cdot a\cdot\nabla\phi + \phi\, b^k\cdot\nabla u_{kn} - \phi\, V^n u_{kn} - u_{kn}\partial_t\phi)dxdt = 0, \tag{5.17}$$
$$\forall\phi \in H^1_0([0,T],H^1(\mathbb{R}^d)) \cap L^\infty((0,T);L^\infty(\mathbb{R}^d)),$$
$$u_{kn}(0) = f \in L^\infty_c(\mathbb{R}^d),\ n\in\mathbb{N}. \tag{5.18}$$

The following statement holds.

Lemma 5.1.9. *There exists a constant $C > 0$ independent of k, n such that*

$$\int_0^T \int_{\mathbb{R}^d} \nabla u_{kn} \cdot a \cdot \nabla u_{kn} dx dt \le C\Big(M_{d+1}(b)\|f\|_1\|f\|_\infty + M_d(V)\|f\|_2^2 + \|f\|_2^2\Big)$$

for all $n \in \mathbb{N}$.

Proof. Applying Theorem 1.2 from [54] we see that the semigroup $\exp(tH_{kn})$ is analytic on $L^1(\mathbb{R}^d)$. Therefore u_{kn} is continuously differentiable on $(0,T]$. For $0 < t_1 < t_2 < T$ and $0 < \varepsilon < (t_2 - t_1)/2$ we define a function $\eta := \eta_{\varepsilon,t_1,t_2} : [0,T] \to [0,1]$ as follows.

$$\eta(t) = \begin{cases} 0 & \text{if } t \in [0,t_1] \cup [t_2,T] \\ \frac{1}{\varepsilon}(t-t_1) & \text{if } t \in (t_1, t_1+\varepsilon) \\ 1 & \text{if } t \in [t_1+\varepsilon, t_2-\varepsilon] \\ \frac{1}{\varepsilon}(t_2-t) & \text{if } t \in (t_2-\varepsilon, t_2) \end{cases}$$

It is easy to verify that ηu_{kn} can be taken as a test function. We make use of (5.17) and obtain that

$$\int_0^T \int_{\mathbb{R}^d} \eta \nabla u_{kn} \cdot a \cdot \nabla u_{kn} dx dt = -\int_0^T \int_{\mathbb{R}^d} \big(u_{kn}\eta\partial_t u_{kn} + u_{kn}^2 \partial_t \eta\big) dx dt$$
$$-\int_0^T \int_{\mathbb{R}^d} b^k \eta \cdot \nabla u_{kn} u_{kn} dx dt + \int_0^T \int_{\mathbb{R}^d} V^n \eta |u_{kn}|^2 dx dt = -I_1 - I_2 + I_3$$

A direct computation yields

$$I_1 = \frac{1}{\varepsilon}\int_{t_1}^{t_1+\varepsilon} \|u_{kn}\|_2^2(t)dt - \frac{1}{\varepsilon}\int_{t_2-\varepsilon}^{t_2} \|u_{kn}\|_2^2(t)dt + \frac{1}{2}\int_{t_1}^{t_2} \eta\partial_t\|u_{kn}\|_2^2(t)dt = I_{11} + I_{12} + I_{13}.$$

The continuity of $\|u_{kn}\|_2(\cdot)$ implies that

$$I_{11} \to \|u_{kn}\|_2^2(t_1) \text{ and } I_{12} \to -\|u_{kn}\|_2^2(t_2) \text{ as } \varepsilon \to 0.$$

From the continuity of $\partial_t\|u_{kn}\|_2(\cdot)$ we infer that

$$I_{13} \to \frac{1}{2}\int_{t_1}^{t_2} \partial_t\|u_{kn}\|_2^2(t)dt = \frac{1}{2}\big(\|u_{kn}\|_2^2(t_2) - \|u_{kn}\|_2^2(t_1)\big) \text{ as } \varepsilon \to 0.$$

Therefore

$$I_1 \le \frac{1}{2}\left(\|u_{kn}\|_2^2(t_2) + \|u_{kn}\|_2^2(t_1)\right) \le C(T)\|f\|_2^2$$

for all $0 < t_1 < t_2 < T$.

In order to estimate I_2 we recall that $\eta(t) \le 1$ and $|b^k| \le |b|$, employ Proposition 5.1.7(i) and estimate (5.16), and apply the Fubini theorem:

$$\begin{aligned}
|I_2| \le & \int_0^T \int_{\mathbb{R}^d} \Big| b^k(x) \cdot \Big(\int_{\mathbb{R}^d} \nabla_x r_{kn}(t,x,y) f(y) dy \Big) \Big(\int_{\mathbb{R}^d} r_{kn}(t,x,y) f(y) dy \Big) \Big| dx dt \\
\le & C\|f\|_\infty \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |b(x)||\nabla_x r_{kn}(t,x,y)||f(y)| dy dx dt \\
\le & C(\gamma)\|f\|_1\|f\|_\infty M_{d+1}(b)\big(1 + M_d(V)\big),
\end{aligned}$$

where the last inequality follows from the estimate $M_{d+1}(|b|, h, \gamma) \le C(\gamma) M_{d+1}(b)$ for all $h > 0$.

Finally we estimate I_3. We employ Proposition 5.1.7(i), Jensen's inequality and the definition of $M_d(V)$, and conclude that

$$\begin{aligned}
|I_3| \le & \int_0^T \int_{\mathbb{R}^d} \Big| V^n(x) \Big(\int_{\mathbb{R}^d} r_{kn}(t,x,y) f(y) dy \Big)^2 \Big| dx dt \\
\le & C_\gamma \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |V(x)| \Gamma_\gamma(t, x - y) |f(y)|^2 dy dx dt \le C\|f\|_2^2 M_d(V).
\end{aligned}$$

Thus

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \eta \nabla u_{kn} \cdot a \cdot \nabla u_{kn} dx dt \le C\Big(M_{d+1}(b)\|f\|_1\|f\|_\infty + M_d(V)\|f\|_2^2 + \|f\|_2^2 \Big)$$

for all $0 < t_1 < t_2 < T$ and $0 < \varepsilon < (t_2 - t_1)/2$. Applying the Fatou lemma we complete the proof. □

Now we are in a position to prove the main result of this section.

Proof of Theorem 5.1.2. Gaussian bounds on r have already been proved before, so it remains to show that r is a weak fundamental solution of equation (5.1), i.e. that for every $f \in L_c^\infty(\mathbb{R}^d)$ and for every $T > 0$ the function

$$u(t,x) := \int_{\mathbb{R}^d} r(t,x,y) f(y) dy, \quad t \in [0,T],\ x \in \mathbb{R}^d,$$

is a weak solution to problem (5.15).

For $k, n \in \mathbb{N}$ let u_{kn} be the same as in Lemma 5.1.9.

Since the semigroup $\exp(tH)$, $t \geq 0$, is a C_0-semigroup on $L^p(\mathbb{R}^d)$ for all $p \in [1, \infty)$, the function $u \in C([0,T]; L^p(\mathbb{R}^d))$ for all $T > 0$. Thus we conclude that

$$\sup_{0 \leq t \leq T} \|u_{kn}(t, \cdot) - u(t, \cdot)\|_p \to 0 \quad \text{as} \quad k, n \to \infty. \tag{5.19}$$

By Proposition 5.1.8 we have

$$\int_0^T \int_{\mathbb{R}^d} |b(x) \cdot \nabla u(t,x)| dx dt \leq C M_{d+1}(b) \big(1 + M_d(V)\big) \|f\|_1.$$

Thus $b \cdot \nabla u \in L^1\big((0,T) \times \mathbb{R}^d\big)$. Employing (5.14) we infer that $Vu \in L^1\big((0,T) \times \mathbb{R}^d\big)$.

Using the definition of u_{kn} we see that

$$\int_0^T \|u_{kn}\|_p^p(t) dt \leq C \|f\|_p^p \int_0^T e^{\omega t} dt \leq C(T) \|f\|_p^p, \quad 1 \leq p < \infty. \tag{5.20}$$

It follows from Lemma 5.1.9, (5.19) and (5.20) that

$$u_{kn} \to u \quad \text{and} \quad \nabla u_{kn} \to \nabla u \quad \text{weakly in} \quad L^2\big((0,T), L^2(\mathbb{R}^d)\big). \tag{5.21}$$

Next we verify the rest of the conditions in (5.17), with u_{kn}, b^k and V^n replaced by u, b and V, respectively. Let ϕ be a test function as in (5.17). Employing (5.21) and (5.17) we see that

$$\int_0^T \int_{\mathbb{R}^d} u \partial_t \phi dx dt = \lim_n \lim_k \int_0^T \int_{\mathbb{R}^d} u_{kn} \partial_t \phi dx dt = \lim_n \lim_k \Bigg[\int_0^T \int_{\mathbb{R}^d} \nabla \phi \cdot a \cdot \nabla u_{k,n} dx dt$$

$$+ \int_0^T \int_{\mathbb{R}^d} \phi b^k \cdot \nabla u_{kn} dx dt - \int_0^T \int_{\mathbb{R}^d} \phi V^n u_{kn} dx dt \Bigg].$$

We conclude from (5.21) that

$$\lim_n \lim_k \int_0^T \int_{\mathbb{R}^d} \nabla \phi \cdot a \cdot \nabla u_{kn} dx dt = \int_0^T \int_{\mathbb{R}^d} \nabla \phi \cdot a \cdot \nabla u dx dt.$$

A straightforward computation yields

$$\int_0^T \int_{\mathbb{R}^d} \phi \big(b^k \cdot \nabla u_{kn} - b \cdot \nabla u\big) dx dt = \int_0^T \int_{\mathbb{R}^d} \phi (b^k - b) \cdot \nabla u_{kn} dx dt$$

$$+ \int_0^T \int_{\mathbb{R}^d} \phi b \cdot (\nabla u_n - \nabla u) dx dt = J_1^{(k,n)} + J_2^{(k,n)}$$

Next we observe that by Proposition 5.1.7(ii)

$$|\nabla u_{kn}(t,x)| \leq C_{\gamma,n} t^{-1/2} \int_{\mathbb{R}^d} \Gamma_\gamma(t,x-y)|f(y)|dy \leq Ct^{-1/2}\|f\|_\infty, \quad t>0,\, x\in\mathbb{R}^N.$$

Hence, for fixed $n \in \mathbb{N}$, the dominated convergence theorem implies that $J_1^{(k,n)} \to 0$ as $k \to \infty$. Repeating the proof of Proposition 5.1.8 with u_n replaced by u_{kn} we infer that $\nabla u_{k_j n_j} \to \nabla u$ in the w^*-topology in $(P\widehat{K}^\gamma_{N+1})^*$. Hence, $J_2^{(k,n)} \to 0$ as $k, n \to \infty$.

In a similar way we show that

$$\lim_n \lim_k \int_0^T \int_{\mathbb{R}^d} \phi(t,x)\big(V^n(x)u_{kn}(t,x) - V(x)u(t,x)\big)dxdt = 0.$$

Thus u is a weak solution of (5.1).

Finally, for $t>0$, $n\in\mathbb{N}$ and $g \in L^2(\mathbb{R}^d)$ we have

$$\left|\int_{\mathbb{R}^d} (u(t,x)-f(x))g(x)dx\right| \leq \left|\int_{\mathbb{R}^d} (u(t,x)-u_{kn}(t,x))g(x)dx\right|$$
$$+\left|\int_{\mathbb{R}^d} (u_{kn}(t,x)-f(x))g(x)dx\right| = I_1^{(k,n)}(t) + I_2^{(k,n)}(t).$$

Indeed, let $\varepsilon > 0$. By (5.19) one can find a pair $k_0, n_0 \in \mathbb{N}$ such that $I_1^{(k_0,n_0)}(t) < \varepsilon/2$. Since $u_{k_0 n_0}$ is a weak solution to (5.17) there is a $\delta_0 > 0$ such that $I_2^{(k_0,n_0)} < \varepsilon/2$ for all $t \in [0,\delta_0]$. Therefore

$$\lim_{t\to 0} \int_{\mathbb{R}^d} u(t,x)g(x)dx = \int_{\mathbb{R}^d} f(x)g(x)dx.$$

Hence, u is a weak solution to (5.15). This completes the proof of existence of a weak fundamental solution. □

Remark. *Let $y_0 \in \mathbb{R}^d$. We claim that the function $u_0(t,x) := r(t,x,y_0)$ is a weak solution of* (5.1) *for all $t>0$ and $x\in\mathbb{R}^d$.*

Indeed, Theorem 5.1.2 implies that

$$u_{0,n}(t,x) := \frac{1}{|B_{\frac{1}{n}}(y_0)|} \int_{B_{\frac{1}{n}}(y_0)} r(t,x,y)dy, \ n\in\mathbb{N},$$

where $B_{\frac{1}{n}}(y_0)$ stands for the ball of radius $\frac{1}{n}$ centered at y_0, are weak solutions of (5.1). *The Lebesgue-Besicovitch differentiation theorem implies that $u_{0,n} \to u_0$*

a.e. We employ the Gaussian upper bound for r and the dominated convergence theorem and see that u_0 is also a weak solution of equation (5.1).

In the next lemma we show that the solution

$$u(t,x) = \int_{\mathbb{R}^d} r(t,x,y)f(y)dy, \quad t > 0,\ x \in \mathbb{R}^d,\ f \in L_c^\infty(\mathbb{R}^d),$$

to problem (5.15) is unique. Obviously this yields uniqueness of the heat kernel r and, therefore, completes the proof of Theorem 5.1.2. We use the same approach as in [53].

Lemma 5.1.10. *(Cf. [53, Lemma 4.7]). Let $T > 0$ and u be a weak solution to the Cauchy problem*

$$\begin{aligned} &u \in C([0,T]; L^2(\mathbb{R}^d)) \cap L^2((0,T); H^1(\mathbb{R}^d)) \\ &\partial_t u = \nabla \cdot a \cdot \nabla u - b \cdot \nabla u - Vu, \\ &b \cdot \nabla u,\ Vu \in L^1\big((0,T) \times \mathbb{R}^d\big), \\ &u(0) = 0, \end{aligned} \tag{5.22}$$

for arbitrary $T > 0$. Then $u \equiv 0$.

Proof. Let $v \in C([0,T]; L^2(\mathbb{R}^d) \cap L^2((0,T); H^1(\mathbb{R}^d))$. First we consider the Cauchy problem

$$\partial_t v - \nabla \cdot a \cdot \nabla v + b^n \cdot \nabla v + V^n v = F, \tag{5.23}$$

$$v(0) = 0,\ \ F \in L^1((0,T) \times \mathbb{R}^d), \tag{5.24}$$

where b^n and V^n, $n \in \mathbb{N}$ are the cut-offs of b and V respectively. It follows from ([53, Lemma 4.6]) that

$$v(t,x) = \int_0^t \int_{\mathbb{R}^d} r_n(s,x,y)F(s,y)dyds,$$

where r_n is the fundamental solution of the equation

$$\partial_t v - \nabla \cdot a \cdot \nabla v + b^n \cdot \nabla v + V^n v = 0.$$

The equation in (5.22) can be rewritten as

$$\partial_t u - \nabla \cdot a \cdot \nabla u + b^n \cdot \nabla u + V^n u = (b^n - b) \cdot \nabla u + (V^n - V)u.$$

By (5.22) the function $F := (b^n - b) \cdot \nabla u + (V^n - V)u \in L^1((0,T) \times \mathbb{R}^d)$. Hence,

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} \Big((b^n(y) - b(y)) \cdot \nabla u(s,y) + (V^n(y) - V(y))u(s,y) \Big) r_n(s,x,y) dyds.$$

Let K be a compact subset of $\mathbb{R}^d$. It is easy to see that

$$\int_0^T \int_K |u(t,x)| dxdt \leq \int_0^T \int_K \int_0^t \int_{\mathbb{R}^d} \Big(\Big| (b^n(y) - b(y)) \cdot \nabla u(s,y) \Big|$$
$$+ \Big| (V^n(y) - V(y))u(s,y) \Big| \Big) r_n(s,x,y) dydsdxdt.$$

Passing to the limit as $n \to \infty$ and using Proposition 5.1.8 and the dominated convergence theorem we conclude that $\int_0^T \int_K |u(t,x)| dxdt = 0$. Since the set K is arbitrary it follows that $u \equiv 0$. □

5.2 Existence and Non-existence of Positive Solutions for a Class of Semi-linear Elliptic Inequalities

We recall that for $0 < R_1 < R_2$ we set $A_{R_2,R_1} := B_{R_2} \setminus \overline{B}_{R_1}$ (see subsection 2.6.4).

5.2.1 Formulation of Main Result

In this section our main concern is existence and non-existence of positive weak super-solutions of the equation

$$\nabla \cdot a \cdot \nabla u - b \cdot \nabla u - Vu + u^p = 0 \quad \text{in } K^c, \tag{5.25}$$

where $p > 1$ and K^c stands for the complement of a compact set K (see subsection 2.6.4 for relevant definitions).

Before stating the second main result of this chapter we modify the conditions on the lower order terms. Namely, we suppose that global conditions of type **(B1)** and **(C1)** are fulfilled outside a large ball, whereas on the whole of $\mathbb{R}^d$ only local conditions are imposed. We also make an additional assumption about the drift coefficient b.

More precisely, we assume that there exist numbers $r > 0$ and $\widehat{R} > 0$ such that $B^c_{\widehat{R}} \subset \Omega$ and

(B1′) $b \in \widehat{K}_{d+1}$ and $M_{d+1}(b,r)$ is sufficiently small (local condition),
$\hat{b} \in \widehat{K}_{d+1,\infty}$ and $M_{d+1}(\hat{b})$ is sufficiently small (global condition);

(C1′) $V \in \widehat{K}_d$ and $M_d(V,r)$ is sufficiently small (local condition),
$\widehat{V} \in \widehat{K}_{d,\infty}$ and $M_d(\widehat{V})$ is sufficiently small (global condition);

(B2′) $|\hat{b}|^2 \le \alpha(-\nabla \cdot a \cdot \nabla)$ in the form sense with a sufficiently small $\alpha > 0$,

where $\hat{b} := b 1\!\!1_{B^c_{\widehat{R}}}$ and $\widehat{V} := V 1\!\!1_{B^c_{\widehat{R}}}$.

Remark. *Condition* (**B2′**) *is satisfied if, for example, the potential* $|\hat{b}|^2 \in \widehat{K}_{d,\infty}$ *and* $M_d\big(|\hat{b}|^2\big)$ *is sufficiently small (see e.g. [71]).*

The main result on the existence and non-existence of positive solutions of semi-linear inequalities in exterior domains reads as follows.

Theorem 5.2.1. *(Cf. Theorem 2.6.4.) Let* $p > 1$ *and* $p_0 := \dfrac{d}{d-2}$*. We assume that conditions* (**A1**), (**A2**), (**B1′**), (**B2′**) *and* (**C1′**) *are fulfilled. Then the following assertions are equivalent:*

(i) $p \le p_0$*;*

(ii) for every compact subset K*, such that* K^c *is connected, there are no non-trivial positive weak super-solutions of* (5.25).

The number p_0 is sometimes called *the critical exponent* for equation (5.25).

5.2.2 Non-existence of Positive Solutions. Case of Sub-critical exponent $p < p_0$

Without loss of generality we can assume that $\widehat{R} = 1$. Let $G = G(x,y)$ stand for the fundamental solution of the equation

$$\nabla \cdot a(x) \cdot \nabla u(x) - b \cdot \nabla u(x) - V(x)u(x) = 0.$$

By Corollary 5.1.3 there exist constants $C_1, C_2 > 0$ such that

$$C_1|x-y|^{2-d} \le G(x,y) \le C_2|x-y|^{2-d} \quad \text{for} \quad x,y \in B_1^c,\ x \ne y. \tag{5.26}$$

In Lemma 5.2.2 below we obtain an a priory estimate for a non-trivial super-solution of the equation $\Lambda u = 0$.

Lemma 5.2.2. *Let u be a non-trivial positive weak solution of the inequality*

$$\nabla \cdot a \cdot \nabla u - b \cdot \nabla u - Vu \leq 0 \quad in \quad B_1^c.$$

We assume that the conditions of Theorem 5.2.1 are fulfilled. Then there exist constants $c_0 > 0$ and $R_0 > 1$ such that

$$u(x) \geq c_0 |x|^{2-d} \quad in \quad B_{R_0}^c.$$

Proof. Since $u \in H^1_{loc}(B_1^c)$ it is clear that $u \in H^1(A_{6,2})$. Thus the Dirichlet problem

$$\nabla \cdot a \cdot \nabla v - b \cdot \nabla v - Vv = 0,$$
$$v \restriction_{\partial B_2} = u \restriction_{\partial B_2} \geq 0, \; v \restriction_{\partial B_4} = u \restriction_{\partial B_4} \geq 0$$

is well-posed. It follows from the maximum principle that $v \geq 0$. Furthermore, by the Harnack inequality we infer that $v > 0$ on $A_{5,3}$.

Next, by the maximum principle (e.g. Proposition 2.6.4), the function $w := u - v \leq 0$. Hence, $u \geq v \geq \bar{c} > 0$ on $A_{5,3}$. In particular, $u \restriction_{\partial B_4} \geq \bar{c} > 0$.

Set $G_0(x) := G(x, 0)$, $x \neq 0$. It follows from (5.26) that there exist constants $\bar{c}_1$, $\bar{c}_2 > 0$ such that

$$\bar{c}_1 \leq G_0(x) \restriction_{\partial B_4} \leq \bar{c}_2.$$

Therefore one can find a constant $c > 0$ such that $u - cG_0 \restriction_{\partial B_4} > 0$. Set $w := u - cG_0$ on B_4^c. One can readily see that $w^- \leq cG_0 \in L^2(B_4^c, |x|^{-2}dx)$. Now by Proposition 2.6.4 we conclude that $w \geq 0$ on B_4^c. Using the lower bound in (5.26) completes the proof. □

The following result is the main tool for proving non-existence of positive weak solutions.

Lemma 5.2.3. *(Cf. [36, Lemma 4.1].) Let Ω be an open bounded subset of $\mathbb{R}^d$. Let $0 \leq W \in L^\infty_{loc}(\Omega)$ and the conditions of Theorem 5.2.1 are satisfied. Then there exists a $\lambda_0 > 0$ such that every positive weak solution of the inequality*

$$\nabla \cdot a \cdot \nabla u - b \cdot \nabla u - Vu + Wu \leq 0 \quad in \quad \Omega \tag{5.27}$$

is identically zero, whenever $W \geq \lambda_0$.

Proof. Let u stand for a positive weak solution of (5.27). Then u is a solution of the inequality

$$\nabla \cdot a \cdot \nabla u - b \cdot \nabla u - Vu \leq 0 \quad \text{in} \quad \Omega.$$

Suppose for a contradiction that u is not identically zero. Then, repeating the first part of the proof of Lemma 5.2.2, we see that $u(x) > 0$ for a.e. $x \in \Omega$.

Let $\theta \in C_0^\infty(\Omega)$. Then the function $\varphi := \theta^2/u$ can be taken as a test function. Integrating by parts we derive the inequality

$$\left\langle \theta\frac{\nabla u}{u} \cdot a \cdot \theta\frac{\nabla u}{u}\right\rangle - 2\left\langle \nabla\theta \cdot a \cdot \theta\frac{\nabla u}{u}\right\rangle - \left\langle b\theta, \theta\frac{\nabla u}{u}\right\rangle - \left\langle V\theta^2\right\rangle + \left\langle W\theta^2\right\rangle \leq 0.$$

Using the Schwarz and Cauchy inequalities and assuming that $W \geq \lambda > 0$ a.e. we infer that

$$\lambda\|\theta\|_2^2 \leq c\Big(\langle\nabla\theta \cdot a \cdot \nabla\theta\rangle + \|b\theta\|_2^2 + \langle|V|\theta^2\rangle\Big).$$

Taking a sequence $(\theta_n)_{n\in\mathbb{N}} \subset C_0^\infty(\Omega)$ such that

$$\frac{\langle\nabla\theta_n \cdot a \cdot \nabla\theta_n\rangle + \|b\theta_n\|_2^2 + \|\,|V|^{\frac{1}{2}}\theta_n\|_2^2}{\|\theta_n\|_2^2} \to \lambda_{1,\Omega},\ n \to \infty,$$

where $\lambda_{1,\Omega}$ is the first eigenvalue of $-\nabla \cdot a \cdot \nabla + |b|^2 + |V|$ on Ω, we obtain the inequality

$$\lambda \leq c\lambda_{1,\Omega},$$

which is clearly a contradiction for all sufficiently large λ. □

Lemmas 5.2.2 and 5.2.3 appear to be sufficient to conclude that equation (5.25) does not have non-trivial positive solutions if $1 < p < p_0$.

Proof of Theorem 5.2.1, (i)⇒(ii). The case $p < p_0$. Let $1 < p < p_0$. Let K be a compact subset of $\mathbb{R}^d$. Let u be a positive weak super-solution of (5.25) on K^c. Without loss of generality we may assume that $K \subset B_1$. Then u is a super-solution of (5.25) in B_1^c. First we show that $u \equiv 0$ in $B_{R_0}^c$. This yields $u \equiv 0$ in K^c by an argument similar to that in the proof of Theorem 2.6.4.

Now we prove that $u \equiv 0$ in $B_{R_0}^c$. It is easy to see that u is a solution of the inequality

$$\nabla \cdot a \cdot \nabla u - b \cdot \nabla u - Vu \leq 0 \ \text{ in } \ B_{R_0}^c,$$

where R_0 is the number determined in Lemma 5.2.2. Lemma 5.2.2 implies that $u(x) \geq c_0|x|^{2-d}$ in $B_{R_0}^c$. Then there is a $\delta > 0$ such that $u^{p-1}(x) \geq \hat{c}|x|^{-2+\delta}$ in $B_{R_0}^c$. One can readily see that u is a solution of the inequality

$$\nabla \cdot a \cdot \nabla u - b \cdot \nabla u - Vu + \hat{c}|x|^{-2+\delta}u \leq 0 \ \text{ in } \ A_{2\rho,\rho}$$

for all $\rho > R_0$. Rescaling the argument as follows $x = \rho x'$, $x' \in [1, 2]$, and observing that $\nabla_x = \frac{1}{\rho}\nabla_{x'}$ we get

$$\frac{1}{\rho^2}\nabla_{x'} \cdot a \cdot \nabla_{x'} u - \frac{1}{\rho} b \cdot \nabla_{x'} u - Vu + \frac{c\rho^\delta |x'|^\delta}{\rho^2 |x'|^{-2}} u \leq 0,$$

or

$$\nabla_{x'} \cdot a \cdot \nabla_{x'} u - \rho b \cdot \nabla_{x'} u - \rho^2 V u + c\rho^\delta |x'|^\delta u \leq 0 \text{ in } A_{2,1}.$$

Finally, we apply Lemma 5.2.3 and conclude that $u \equiv 0$ in $B^c_{R_0}$ since $\rho^\delta \to \infty$ as $\rho \to \infty$. □

5.2.3 Non-existence of Positive Solutions. Case of Critical Exponent $p = p_0$

The case $p = p_0$ turns out to be more delicate. We begin with an investigation of the equation

$$\nabla \cdot a(x) \cdot \nabla v(x) + \nabla(b(x)v(x)) - V(x)v(x) = 0, \quad x \in \mathbb{R}^d. \tag{5.28}$$

The following result holds.

Lemma 5.2.4. *Let the conditions of Theorem 5.2.1 be fulfilled. Then there exist a solution v of equation* (5.28) *and constants $0 < c_1 \leq c_2 < \infty$, such that*

$$c_1 \leq v(x) \leq c_2.$$

Proof. Let $R > 0$. We start with the problem

$$\nabla \cdot a \cdot \nabla w = -\nabla(b^n(w+1)) + V^n(w+1) \text{ in } B_R,$$
$$w \restriction_{\partial B_R} = 0,$$

where $(b^n)_{n\in\mathbb{N}}$ and $(V^n)_{n\in\mathbb{N}}$ are the cut-offs of b and V respectively. It is well-known (see e.g., [28, Ch. 8]) that the above problem has a weak solution $w_n \in H^1_0(B_R) \cap L^\infty(B_R)$. Note that $v_{n,R} := w_n + 1$ is a weak solution of equation (5.28) on B_R, with b and V replaced by b^n and V^n respectively. We make use of the representation formula for the solution w_n and the Fubini theorem, and obtain that

$$\begin{aligned} w_n(x) &= -\int_{B_R} G_R(x,y)\big(\nabla_y(b^n(y)(w_n(y)+1)) - V^n(y)(w_n(y)+1)\big)dy \\ &= \int_0^\infty \int_{B_R} \Big(\nabla_y p_R(t,x,y) \cdot b^n(y) - p_R(t,x,y)V^n(y)\Big)(w_n(y)+1)dydt, \end{aligned} \tag{5.29}$$

where G_R and p_R stand for the Green functions of the equations $\nabla \cdot a \cdot \nabla u = 0$ and $\nabla \cdot a \cdot \nabla u = \partial_t u$ in the ball B_R, respectively (here we have used the equality $G_R(x,y) = \int_0^\infty p_R(t,x,y)dt$). It is clear that the estimates of type (5.2) and (5.3) can be derived for p_R and ∇p_R respectively. Hence, employing the Fubini theorem we conclude that

$$\begin{aligned}
|w_n(x)| &\leq \int_0^\infty \int_{B_R} \Big(|\nabla_y p_R(t,x,y)||b^n(y)| + p_R(t,x,y)|V^n(y)|\Big)\big(|w_n(y)|+1\big)dydt \\
&\leq (1+\omega_n)C_\beta \int_0^\infty \int_{\mathbb{R}^d} \Big(|b(y)|t^{-1/2}\Gamma_\beta(t,x-y) + |V(y)|\Gamma_\beta(t,x-y)\Big)dydt \\
&\leq (1+\omega_n)C_\beta\Big(M_{d+1}(b) + M_d(V)\Big),
\end{aligned}$$

where $\omega_n := \sup_{y\in B_R}|w_n(y)|$. This yields

$$\omega_n \leq (1+\omega_n)C_\beta\Big(M_{d+1}(b) + M_d(V)\Big).$$

Now assuming that $M_{d+1}(b)$ and $M_d(V)$ are sufficiently small we conclude that $\omega < 1$. Therefore there exist constants c_1, $c_2 > 0$, independent of R and n, such that $c_1 \leq v \leq c_2$.

Next we observe that for every compact set $K \subset \mathbb{R}^d$ there exists a constant $C_K > 0$, independent of R and n, such that

$$\|\mathbb{1}_K \nabla v_{n,R}\|_2^2 \leq C_K$$

for all $n \in \mathbb{N}$ and $R > R'+1$, with $B_{R'} \supset K$. Hence, there is a function $v \in H^1_{loc}(\mathbb{R}^d)$ such that $v_{n_k,R_m} \to v$ in $H^1_{loc}(\mathbb{R}^d)$. It is easy to verify that v is a weak solution of (5.28). □

Next we establish an additional estimate for the solution to the problem

$$\begin{aligned}
&\nabla \cdot a \cdot \nabla v - b \cdot \nabla v - Vv + \nu|x|^{-2}v = 0 \ \text{ in } \ B_R^c \\
&v \restriction_{\partial B_R} > 0,
\end{aligned} \tag{5.30}$$

with arbitrary $R > 1$ and some small $\nu > 0$.

Lemma 5.2.5. *(Cf. Lemma 2.6.7). Let the conditions of Theorem 5.2.1 be fulfilled. Then there exist a unique solution v of* (5.30) *and the constants $C_0 > 0$ and $R_1 > R$ such that*

$$v(x) \geq C_0|x|^{2-d}\log|x| \quad \textit{for all} \quad x \in B_{R_1}^c.$$

Proof. It follows from the Hardy inequality that

$$\nu\langle|x|^{-2}\varphi^2\rangle \le \nu c_H\|\nabla\varphi\|_2^2 \le \nu c_H\zeta\langle\nabla\varphi\cdot a\cdot\nabla\varphi\rangle.$$

Hence, Proposition 2.6.5 implies that there exists a unique solution to (5.30), provided ν is sufficiently small. We divide the rest of the proof into two steps.

Step 1. First we establish an estimate of the Caccioppoli type. Namely, for arbitrary $\rho > 2$

$$\int_{A_{2\rho,\rho}} |\nabla v(x)|^2 dx \le c m_\rho^2 \rho^{d-2}, \tag{5.31}$$

where v is the solution of (5.30).

Let $\rho > 2R$. We choose a function $\theta \in C_0^\infty(\mathbb{R}^d)$ in such way that $0 \le \theta \le 1$, $\operatorname{supp}\theta \subset A_{\frac{5\rho}{2},\frac{\rho}{2}}$ and $\theta = 1$ in $A_{2\rho,\rho}$. We multiply equation (5.30) by the test function $\theta^2 v$ and integrate over $\mathbb{R}^d$. In order to shorten our notation we write $\|\varphi\|_2^2$ in place of $\langle\nabla\varphi\cdot a\cdot\nabla\varphi\rangle$. We get

$$\begin{aligned}
&-\langle\nabla v\cdot a\cdot\nabla(\theta^2 v)\rangle - \langle b\cdot\nabla v,\, \theta^2 v\rangle - \langle V\theta^2 v^2\rangle + \nu\langle|x|^{-2}\theta^2 v^2\rangle\\
&-\big\langle\theta\nabla v\cdot a\cdot\theta\nabla v\big\rangle - 2\big\langle\theta\nabla v\cdot a\cdot v\nabla\theta\big\rangle - \big\langle b\theta v,\, \theta\nabla v\big\rangle - \langle V\theta^2 v^2\rangle + \nu\langle|x|^{-2}\theta^2 v^2\rangle\\
&= -\,\langle\nabla(\theta v)\cdot a\cdot\nabla(\theta v)\rangle - \langle b\theta v,\, \nabla(\theta v)\rangle + \langle b\theta v,\, v\nabla\theta\rangle\\
&+\langle v\nabla\theta\cdot a\cdot v\nabla\theta\rangle - \langle V\theta^2 v^2\rangle + \nu\langle|x|^{-2}\theta^2 v^2\rangle = 0.
\end{aligned}$$

Next we make use of assumptions (**B2′**) and (**C1′**), apply the Schwarz and Cauchy inequalities and obtain the estimate

$$\|\nabla(\theta v)\|_2^2 \le \bar{c}\langle|x|^{-2}\theta^2 v^2\rangle + \hat{c}\|v\nabla\theta\|_2^2. \tag{5.32}$$

We set $m_\rho := \inf_{|x|=\rho} v(x)$. A simple rescaling argument and the Harnack inequality imply that there exist constants c, $C > 0$, independent of ρ, such that $cm_\rho \le v(x) \le Cm_\rho$ for all $x \in A_{\frac{5\rho}{2},\frac{\rho}{2}}$. Estimate (5.32) and the definition of θ imply that

$$\int_{A_{2\rho,\rho}} |\nabla v(x)|^2 dx \le \bar{c} \int_{A_{\frac{5\rho}{2},\frac{\rho}{2}}} \frac{v^2(x)}{|x|^2} dx + \hat{c}\Big(\int_{A_{\rho,\frac{\rho}{2}}} + \int_{A_{\frac{5\rho}{2},2\rho}}\Big) v^2(x)|\nabla\theta(x)|^2 dx$$

Using the Harnack inequality we estimate each integral by $cm_\rho^2\rho^{d-2}$. Hence, (5.31) is proved.

Step 2. We choose a function $\varphi \in C_0(\mathbb{R}^d)$ in such way that $0 \le \varphi \le 1$, $\varphi = 1$ on $A_{\rho,2R}$, $\operatorname{supp}\varphi \subset A_{2\rho,\frac{3R}{2}}$ and $|\nabla\varphi| \le 2/\rho$. It follows from Lemma 5.2.4 that there is a solution v_1 of the equation

$$\nabla \cdot a \cdot \nabla v + \nabla(bv) - Vv = 0 \quad \text{in} \quad B_{2\rho},$$

such that $c_1 \le v_1 \le c_2$. Similar to Step 1 we conclude that

$$\int\limits_{A_{2\rho,\rho}} |\nabla v_1(x)|^2 dx \le \hat{C}\rho^{d-2} \tag{5.33}$$

with some $\hat{C} > 0$. We multiply our equation by the test function $\varphi_1 := v_1\varphi$. A straightforward computation and the definition of v_1 imply that

$$\begin{aligned} &-\langle \nabla v \cdot a \cdot \nabla\varphi_1\rangle - \langle b \cdot \nabla v,\, \varphi_1\rangle + \nu\langle |x|^{-2}\varphi_1 v\rangle - \langle Vv\varphi_1\rangle \\ =&-\langle \nabla v \cdot a \cdot v_1\nabla\varphi\rangle + \langle v,\, \nabla v_1 \cdot a \cdot \nabla\varphi\rangle + \langle bv,\, v_1\nabla\varphi\rangle + \nu\langle |x|^{-2}\varphi_1 v\rangle = 0. \end{aligned}$$

Hence,

$$\nu\langle |x|^{-2}\varphi_1 v\rangle = \langle \nabla v \cdot a \cdot v_1\nabla\varphi\rangle - \langle v,\, \nabla v_1 \cdot a \cdot \nabla\varphi\rangle - \langle bv,\, v_1\nabla\varphi\rangle. \tag{5.34}$$

Making use of Lemmas 5.2.4 and 5.2.2 we estimate the left-hand side of (5.34) below.

$$\nu\langle |x|^{-2}\varphi_1 v\rangle \ge \nu c_1 \int\limits_{A_{\rho,2R}} \frac{v(x)}{|x|^2}dx \ge \hat{c} \int\limits_{A_{\rho,2R}} |x|^{-d}dx \ge \bar{c}\log\rho.$$

Before estimating the right-hand side of (5.34) we note that by Step 1

$$\begin{aligned} &\left|\int\limits_{A_{2\rho,\rho}} v_1\nabla v \cdot a \cdot \nabla\varphi dx\right| \le c_2 \int\limits_{A_{2\rho,\rho}} |\nabla v \cdot a \cdot \nabla\varphi| dx \\ \le &c_2\zeta\rho^{-1}\Big(\int\limits_{A_{2\rho,\rho}} |\nabla v(x)|^2 dx\Big)^{\frac{1}{2}} Vol(B_\rho)^{\frac{1}{2}} \\ \le &\bar{C}m_\rho\rho^{d/2-2}\rho^{d/2} = \bar{C}m_\rho\rho^{d-2}. \end{aligned}$$

Similarly, by (5.33) we infer that

$$\begin{aligned} \left|\int\limits_{A_{2\rho,\rho}} v\nabla v_1 \cdot a \cdot \nabla\varphi dx\right| &\le \zeta\rho^{-1}\Big(\int\limits_{A_{2\rho,\rho}} v^2(x)dx\Big)^{\frac{1}{2}}\Big(\int\limits_{A_{2\rho,\rho}} |\nabla v_1(x)|^2 dx\Big)^{\frac{1}{2}} \\ &\le \bar{C}m_\rho\rho^{d-2}. \end{aligned}$$

Before estimating the last integral in the right-hand side of (5.34) we note that by the Cauchy inequality

$$\int_{A_{\frac{9\rho}{4},\frac{3\rho}{4}}} |\nabla(v(x)\psi(x))|^2 dx \le (1+\delta) \int_{A_{\frac{5\rho}{2},\frac{\rho}{2}}} |\psi(x)\nabla v(x)|^2 dx$$
$$+\left(1+\frac{1}{\delta}\right) \int_{A_{\frac{5\rho}{2},\frac{\rho}{2}}} |v(x)\nabla\psi(x)|^2 dx \le \bar{C} m_\rho \rho^{d-2},$$

where $\psi \in C_0^\infty(\mathbb{R}^d)$, $0 \le \psi \le 1$, $\operatorname{supp}\psi \in A_{\frac{9\rho}{4},\frac{3\rho}{4}}$, $\psi = 1$ on $A_{2\rho,\rho}$ and $|\nabla\psi| \le 8/\rho$. Making use of condition (**B2′**) we conclude that

$$\left|\int_{A_{2\rho,\rho}} vv_1 b\cdot\nabla\varphi dx\right| \le c_2 \int_{A_{2\rho,\rho}} |b(x)v(x)||\nabla\varphi(x)|dx$$
$$\le C\rho^{-1} \int_{A_{\frac{9\rho}{4},\frac{3\rho}{4}}} |b(x)v(x)\psi(x)|dx \le \hat{C}\rho^{d/2-1}\left(\int_{A_{\frac{9\rho}{4},\frac{3\rho}{4}}} |b(x)v(x)\psi(x)|^2 dx\right)^{\frac{1}{2}}$$
$$\le \hat{C}\rho^{d/2-1}\left(\int_{A_{\frac{9\rho}{4},\frac{3\rho}{4}}} |\nabla(v(x)\psi(x))|^2 dx\right)^{\frac{1}{2}} \le \bar{C} m_\rho \rho^{d-2}.$$

We also observe that there exists a constant C independent of ρ such that

$$\left|\int_{A_{2R,\frac{3R}{2}}} \big(v_1\nabla v\cdot a\cdot\nabla\varphi - v\nabla v_1\cdot a\cdot\nabla\varphi - vv_1 b\cdot\nabla\varphi\big)dx\right| \le C.$$

Now it is readily seen that the right-hand side of (5.34) is estimated above by

$$\left|\int_{A_{2R,\frac{3R}{2}}} \big(v_1\nabla v\cdot a\cdot\nabla\varphi - v\nabla v_1\cdot a\cdot\nabla\varphi - vv_1 b\cdot\nabla\varphi\big)dx\right|$$
$$+\left|\int_{A_{2\rho,\rho}} v_1\nabla v\cdot a\cdot\nabla\varphi dx\right| + \left|\int_{A_{2\rho,\rho}} v\nabla v_1\cdot a\cdot\nabla\varphi dx\right| + \left|\int_{A_{2\rho,\rho}} vv_1 b\cdot\nabla\varphi dx\right|$$
$$\le C + 3\bar{C} m_\rho \rho^{d-2}$$

Substituting the obtained inequalities into (5.34) we infer that

$$C + 3\bar{C} m_\rho \rho^{d-2} \ge \bar{c}\log\rho.$$

Therefore there exist ρ_0 and C_0 such that for all $\rho \ge \rho_0$

$$m_\rho \ge C_0 \rho^{2-d}\log\rho.$$

Finally the Harnack inequality implies that $v(x) \geq cC_0|x|^{2-d}\log|x|$ for all $x \in B^c_{\rho_0}$. □

Now we are ready to complete the proof of the (i)⇒(ii) implication in Theorem 5.2.1.

Proof of Theorem 5.2.1, (i)⇒(ii). The case $p = p_0$. Now let u be a positive weak solution of (5.25) with $p = p_0$. Similar to the case $p < p_0$ we infer that $u(x) \geq c_0|x|^{2-d}$ for all $x \in B^c_{R_0}$. Let $\kappa := c_0^{\frac{2}{d-2}} \wedge \nu$, where ν was determined in Lemma 5.2.5. Then u is a solution to the problem

$$\nabla \cdot a \cdot \nabla u - b \cdot \nabla u - Vu + \kappa|x|^{-2}u \leq 0 \ \text{ in } \ B^c_{R_0},$$
$$u \restriction_{\partial B_{R_0}} > 0.$$

Let $v \in \mathcal{H}_{R_0}$ stand for the solution to

$$\nabla \cdot a \cdot \nabla v - b \cdot \nabla v - Vv + \kappa|x|^{-2}v \leq 0 \ \text{ in } \ B^c_{R_0},$$
$$v \restriction_{\partial B_{R_0}} = u \restriction_{\partial B_{R_0}} > 0.$$

We set $w := u - v$. Then we have

$$\nabla \cdot a \cdot \nabla w - b \cdot \nabla w - Vw \leq 0,$$
$$w \restriction_{\partial B_{R_0}} = 0.$$

The maximum principle (Proposition 2.6.4) yields $w \geq 0$, i.e. $u \geq v$. Hence, Lemma 5.2.5 yields

$$u(x) \geq C_0|x|^{2-d}\log|x| \ \text{ in } \ B^c_{R_1}.$$

Similar to the proof in the case $p < p_0$, the last estimate and Lemma 5.2.3 imply that $u \equiv 0$ in $B^c_{R_0}$, which, in turn, yields $u \equiv 0$ in K^c. □

5.2.4 Existence of Positive Solutions

Proposition 5.2.6 below completes the proof of Theorem 5.2.1.

Proposition 5.2.6. *(Cf. Proposition 2.6.8). Let the conditions of Theorem 5.2.1 be fulfilled. If $p > p_0$, then one can find a number $\mu > 0$ such that there exists a positive solution $u \in \mathcal{H}_1$ of equation (5.25), with $u \restriction_{\partial B_1} = \mu$.*

Proof. Let $p > p_0$. For $R > 1$ we define the mapping $T : S_R \to L^2(A_{R,1})$ by $Tf := u_R$, $f \in S_R$, where $S_R := \{f \in L^2(A_{R,1}) \mid 0 \le f(x) \le c_0|x|^{2-d}\}$ and u_R is the weak positive solution to the problem

$$\begin{aligned} &\nabla \cdot a \cdot \nabla u - b \cdot \nabla u - Vu + f^{p-1}u = 0 \ \text{ in } \ A_{R,1}, \\ &u \restriction_{\partial B_1} = \mu > 0, \ u \restriction_{\partial B_R} = 0, \end{aligned} \tag{5.35}$$

with $f \in S_R$ (existence and uniqueness of u_R follows from Proposition 2.6.5, provided the constant c_0 is sufficiently small).

It is easy to check that the S_R is a closed, convex subset of $L^2(A_{R,1})$. The next goal is to prove that $T(S_R) \subset S_R$ and $T(S_R)$ is compact.

We set $\varepsilon := (d-2)(p-p_0) > 0$, $\bar{b} := b\mathbb{1}_{B_1^c}$, $\overline{V} := V\mathbb{1}_{B_1^c}$ and $\overline{W} := c|x|^{-2-\varepsilon}\mathbb{1}_{B_1^c}$, and denote by $G_\varepsilon = G_\varepsilon(x,y)$ the fundamental solution of the equation

$$\nabla \cdot a \cdot \nabla v - \bar{b} \cdot v - \overline{V}v + \overline{W}v = 0.$$

The potential $\overline{W}$ is known to belong to $\widehat{K}_{d,\infty}$ (see e.g. [83, Prop. 3.1]). Choosing $c > 0$ in such way that $|\bar{b}|^2 + |\overline{V}| + \overline{W} < -\nabla \cdot a \cdot \nabla$, and employing Corollary 5.1.3 we conclude that there are constants $C_1, C_2 > 0$ such that

$$C_1|x-y|^{2-d} \le G_\varepsilon(x,y) \le C_2|x-y|^{2-d} \ \text{ for all } \ x \ne y.$$

We set $v(x) := \bar{c}G_\varepsilon(x,0)$, $x \in B_1^c$, and $w_R(x) := v(x) - u_R(x)$ in $A_{R,1}$. A direct computation yields

$$\begin{aligned} &\Lambda w_R + c|x|^{-2-\varepsilon}w_R = (f^{p-1} - c|x|^{-2-\varepsilon})u_R \le 0, \ \text{ provided } \ c_0^{p-1} \le c, \\ &w_R \restriction_{\partial B_1} = \bar{c}\,G_\varepsilon(1,0) - \mu \ge \bar{c}\,C_1 - \mu > 0, \ \text{ whenever } \ \mu < \bar{c}\,C_1, \\ &w_R \restriction_{\partial B_R} = \bar{c}\,G_\varepsilon(R,0) \ge \bar{c}\,C_1 R^{2-d} > 0. \end{aligned}$$

We observe that $w_R^- \le u_R$, so $w_R^- \in L^2(B_1^c, |x|^{-2}dx)$. Taking $c > 0$ to be small enough and employing Proposition 2.6.4 we conclude that $w_R \ge 0$. Hence,

$$u_R(x) \le v(x)\,\bar{c}\,G_\varepsilon(x,0) \le \bar{c}\,C_2|x|^{2-d}.$$

Choosing $\bar{c}$ in such way that $\bar{c}C_2 \le c_0$ we infer that $u_R \in S_R$. Thus $T(S_R) \subset S_R$. We shall see below that $u_R \in H^1(A_{R,1})$, so the set $T(S_R) \subset H^1(A_{R,1}) \cap S_R$ is compact in $L^2(A_{R,1})$.

Next we observe that the mapping T is continuous. Indeed, let $(f_n)_{n\in\mathbb{N}} \subset S_R$ be such that $f_n \to f \in S_R$ in $L^2(A_{R,1})$. For $n \in \mathbb{N}$ let u_n stand for the solution to

(5.35) with f replaced by f_n. It is easy to see that $w_n := u - u_n$ is a solution to the problem

$$\Lambda w_n + f^{p-1} w_n + u_n(f^{p-1} - f_n^{p-1}) = 0,$$
$$w_n \restriction_{\partial B_1} = w_n \restriction_{\partial B_R} = 0.$$

We note that w_n can be taken as a test function. Integrating over $A_{R,1}$, using form-boundedness of $|b|^2$ and V and applying the Hardy inequality we conclude that

$$\|\nabla w_n\|_2^2 \leq C\big|\langle u_n w_n,\, f^{p-1} - f_n^{p-1}\rangle\big|.$$

It follows from the dominated convergence theorem that $\|\nabla w_n\|_2 \to 0$. Hence, $w_n \to 0$ in $L^2(A_{R,1})$.

Hence, by the Schauder fixed-point theorem (Theorem 2.6.9) there exists a function $u_R^* \in S_R$ such that $T(u_R^*) = u_R^*$, i.e. u_R^* is a solution to the problem

$$\nabla \cdot a \cdot \nabla u_R - b \cdot \nabla u_R - V u_r + u_R^p = 0,$$
$$u_R \restriction_{\partial B_1} = \mu > 0, \;\; u_R \restriction_{\partial B_R} = 0.$$

Further on we denote u_R^* by u_R. Let $\phi \in C_0^1(\mathbb{R}^d)$, $\phi \restriction_{\partial B_1} = \mu$ and $\operatorname{supp}\phi \subset A_{\frac{3}{2},\frac{1}{2}}$. Since $u_R - \phi \restriction_{\partial B_1} = u_R - \phi \restriction_{\partial B_R} = 0$ we can take $u_R - \phi$ as a test function. We have

$$\begin{aligned} 0 &= \langle \nabla \cdot a \cdot \nabla u_R - b \cdot \nabla u_R + u_R^p,\, u_R - \phi\rangle \\ &= -\|\nabla u_R\|_2^2 + \langle \nabla u_R,\, \nabla\phi\rangle - \langle b \cdot \nabla u_R,\, u_R\rangle + \langle b \cdot \nabla u_R,\, \phi\rangle + \langle u_R^{p+1}\rangle - \langle u_R^p \phi\rangle. \end{aligned}$$

Using conditions (**B2′**), (**C1′**) and the Cauchy inequality we conclude that

$$\|\nabla u_R\|_2^2 \leq c\big(\|\nabla\phi\|_2^2 + \langle u_R^{p+1}\rangle\big) \leq C,$$

with C independent of R. A direct computation shows that

$$\left\|\frac{u_R}{r}\right\|_2^2 \leq c_0^2 c_d \int_1^R r^{2-2d} r^{d-1}\, dr \leq C.$$

Therefore the set $(u_R)_{R>1}$ is uniformly bounded in $\mathcal{H}_1$. So there exist a sequence $(u_{R_n})_{n\in\mathbb{N}}$, $R_n \to \infty$, and a function $u \in \mathcal{H}_1$ such that $u_{R_n} \to u$ weakly in $\mathcal{H}_1$. It is easy to check that u is a solution of (5.25). □

5.2.5 Sharpness of Results

In this subsection we give two examples which show that the results stated in Theorem 5.2.1 are sharp.

Let $W(x) = a|x|^\alpha \wedge 1$, $x \in \mathbb{R}^d$, $a > 0$, $\alpha \in \mathbb{R}$. As was mentioned above (see subsection 5.1.1) the potential $W \in \widehat{K}_{d,\infty}$ iff $\alpha < -2$, and $W \in \widehat{K}_{d+1,\infty}$ iff $\alpha < -1$.

The first example shows that Theorem 5.2.1 is no longer true if assumptions (**B1′**)-(**B2′**) are not fulfilled.

Example 5.2.7. *For every $p > 1$ one can find a number $c \in \mathbb{R}$ such that the inequality*

$$\Delta u + \frac{cx}{|x|^2}\nabla u + u^p \leq 0 \tag{5.36}$$

has a positive solution in B_1^c.

We claim that there exist $a > 0$ and $\alpha < 0$ such that the function $u(x) := a|x|^\alpha$, $x \in B_1^c$, is a solution of (5.36). Indeed, substituting u into (5.36) we get

$$\Delta u + \frac{cx}{|x|^2}\nabla u + u^p = a|x|^{\alpha-2}(\alpha(\alpha-1) + \alpha(d-1) + c\alpha) + a^p|x|^{\alpha p} \leq 0.$$

Choosing $\alpha = \dfrac{2}{1-p}$ we obtain the inequality

$$\alpha(\alpha-1) + \alpha(d-1) + c\alpha + a^{p-1} \leq 0,$$

which always holds, provided

$$\alpha(\alpha-1) + \alpha(d-1) + c\alpha < 0, \quad \text{or} \quad \alpha + d - 2 + c > 0.$$

Hence, for every $p > 1$ one can find a number $c \in \mathbb{R}$ such that the estimate

$$c + d - 2 + \frac{2}{1-p} > 0$$

is satisfied.

Let $W(x) = a|x|^\alpha \log^\beta |x| \wedge 1$, $x \in \mathbb{R}^d$, $a > 0$, $\alpha, \beta \in \mathbb{R}$. We recall that $W \in \widehat{K}_{d,\infty}$ if $\alpha = -2$ and $\beta < -1/2$, and $W \in \widehat{K}_{d+1,\infty}$ if $\alpha = -1$ and $\beta < -1$.

In the following example the critical exponent p_0 belongs to the case of existence of positive weak super-solutions.

Example 5.2.8. *One can find a number $c \in \mathbb{R}$ such that the inequality*

$$\Delta u + \frac{cx}{|x|^2 \log |x|} \nabla u + u^{\frac{d}{d-2}} \leq 0 \tag{5.37}$$

has a positive solution in B_1^c.

Let u be a solution of (5.37). We assume that $u(x) = \Phi(r) = ar^\alpha \log^\beta r$, $x \in B_1^c$. Then Φ satisfies the following one-dimensional inequality:

$$\Phi'' + \frac{d-1}{r}\Phi' + \frac{c}{r \log r}\Phi' + \Phi^{p_0} \leq 0. \tag{5.38}$$

A direct computation implies that if $\alpha = \beta = 2 - d$, then one can find a constant c such that Φ is a solution of (5.38). Hence, (5.37) has a positive solution $u(x) = a|x|^{2-d} \log^{2-d} |x|$, B_1^c, $a > 0$.

We note that in Example 5.2.8 the drift coefficient b satisfies condition (**B2′**) but not (**B1′**). Thus the sharpness of (**B1′**) is established.

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